第四届“刘徽杯”数学竞赛

第一天 (2021 年 11 月 13 日)

第 1 题 在三角形 ABC 中，M, N 分别是边 AB, AC 的中点. 直线 BN, CM 分别再次交 $\odot(ABC)$ 于 E, F. 直线 EF 分别交 AB, AC 于 S, T. 圆 $\odot(FSM)$, $\odot(ETN)$ 分别再次交 $\odot(ABC)$ 于 U, V. 证明: $|AU| = |AV|$.
这里 $\odot(XYZ)$ 表示三角形 XYZ 的外接圆.

第 2 题 求所有满足 $\sqrt{xy} + \sqrt{uv}$ 为有理数且

$$\left|\frac{x}{9} - \frac{y}{4}\right| = \left|\frac{u}{3} - \frac{v}{12}\right| = uv - xy$$

的正整数组 (x, y, u, v).

第 3 题 对于正整数 n, 记 $f(n)$ 为满足以下条件的整数序列 $\{a_i\}_{i=1}^{l}$ 的最大项数:

- $0 \le a_1 < a_2 < \cdots < a_l \le n$.
- 当 $1 \le i \le j \le k \le l$, $1 \le i' \le j' \le k' \le l$, 且 $(i, j, k) \ne (i', j', k')$ 时 $a_i + a_j + a_k \ne a_{i'} + a_{j'} + a_{k'}$.

证明:

$$f(n) \le \left\lfloor 4n\left(1 - \frac{1}{6\log_2^2 n}\right)\right\rfloor^{1/3} + 7,$$

其中 $\lfloor x \rfloor$ 表示不超过实数 x 的最大整数.

Fourth Liu Hui Cup Mathematical Olympiad

Day 1 (November 13, 2021)

Problem 1. In a given triangle ABC, M and N are the midpoints of sides AB and AC, respectively. Lines BN, CM meet $\odot(ABC)$ again at E, F, respectively. Line EF intersects AB and AC, respectively, at S and T. Circles $\odot(FSM)$ and $\odot(ETN)$ meet $\odot(ABC)$ again at U and V, respectively. Show that $|AU| = |AV|$. Here $\odot(XYZ)$ denotes the circumcircle of triangle XYZ.

Problem 2. Find all quadruples (x, y, u, v) of positive integers such that $\sqrt{xy} + \sqrt{uv}$ is rational and

$$\left|\frac{x}{9} - \frac{y}{4}\right| = \left|\frac{u}{3} - \frac{v}{12}\right| = uv - xy.$$

Problem 3. For each positive integer n, let $f(n)$ denote the largest integer l such that there exists an l-term sequence of integers $\{a_i\}_{i=1}^{l}$ satisfying

- $0 \le a_1 < a_2 < \cdots < a_l \le n$;
- $a_i + a_j + a_k \ne a_{i'} + a_{j'} + a_{k'}$ for $1 \le i \le j \le k \le l$ and $1 \le i' \le j' \le k' \le l$ with $(i, j, k) \ne (i', j', k')$.

Show that

$$f(n) \le \left\lfloor 4n\left(1 - \frac{1}{6\log_2^2 n}\right)\right\rfloor^{1/3} + 7,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.