第一届“刘徽杯”数学竞赛

第一天 (2018 年 10 月 5 日)

第 1 题 设 n 是一个给定的大于 2 的整数. 在单位圆的内接正 n 边形 $A_1A_2\ldots A_n$ 内 (含边界) 有一动点 P, 求 $\prod_{k=1}^{n}|PA_k|$ 的最大值.

第 2 题 设 n 为大于 2018 的正整数. 证明: 存在正整数 a, b, 同时满足

(1) $n=a-b$;

(2) a 的不同素因数构成的集合恰好比 b 的不同素因数的集合多 1 个元素, 并且多的那个素因数小于 n.

第 3 题 设 n 为正整数. 若正整数 A 的十进制表示不含数字 2, 0, 1, 8, 且 A 的任何相邻两位数字按原顺序所成的两位数都是素数, 称这样的正整数 A 为“丰收数”. 例如 6, 47, 379 都是“丰收数”. 以 a_n 表示不超过 n 位的“丰收数”的个数, 易得 $a_1=6$, $a_2=15$, $\ldots$.

(1) 求证: 数列 $\{a_n\}$ 的通项公式为

$$a_n=\left[P+(-1)^nQ\right]\cdot 2^{(n-4)/2}-16,\qquad (n=1,2,3,\ldots),$$

其中 $P=31+22\sqrt{2}$, $Q=31-22\sqrt{2}$.

(2) 十进制表示中不含数字 4, 5, 6 且能被 11 整除的“丰收数”称为“中秋丰收数”. 设全体 n 位“中秋丰收数”的个数为 c_n, 它们的和为 S_n, 其中 $n=1,2,3,\ldots$. 求证:

$$\frac{S_{924}}{c_{924}}=\frac{13(10^{924}-1)}{18}.$$

First Liu Hui Cup Mathematical Olympiad

Day 1 (October 5, 2018)

Problem 1. Let n be a given integer greater than two. A point P is inside the regular n-gon $A_1A_2\ldots A_n$ (including its boundary) that is inscribed in the unit circle. Find the maximum value of $\prod_{k=1}^{n}|PA_k|$ as P varies.

Problem 2. Let n be a positive integer greater than 2018. Prove that there exist positive integers a and b satisfying both the following:
(1) $n = a - b$;
(2) the set of distinct prime factors of a has exactly one additional element compared to that of b, with the additional element being less than n.

Problem 3. Let n be a positive integer. A positive integer A is called a *harvest number* if its decimal (i.e., base-ten) representation does not contain digits 2, 0, 1, or 8, and any two consecutive digits of A (preserving their original order) form a 2-digit prime number. For example, 6, 47, and 379 are harvest numbers. Let a_n be the number of the harvest numbers that have at most n decimal digits. Then $a_1 = 6$, $a_2 = 15$, $\ldots$.
(1) Show that the general term of the sequence $\{a_n\}$ is

$$a_n = \left[P + (-1)^n Q\right] \cdot 2^{(n-4)/2} - 16, \qquad (n = 1, 2, 3, \ldots),$$

where $P = 31 + 22\sqrt{2}$ and $Q = 31 - 22\sqrt{2}$.
(2) A *Mid-Autumn harvest number* is a harvest number divisible by 11 and does not contain digits 4, 5, or 6 in its decimal representation. Let c_n be the total number of n-digit Mid-Autumn harvest numbers and S_n the sum of all these n-digit Mid-Autumn harvest numbers (for $n = 1, 2, 3, \ldots$). Prove that

$$\frac{S_{924}}{c_{924}} = \frac{13(10^{924} - 1)}{18}.$$

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第二天 (2018 年 10 月 6 日)

第 4 题 设点 M 是三角形 ABC 的外接圆 Γ 上弧 BC (不含点 A) 的中点, 点 J 是三角形 ABC 的 B-旁心, 直线 JM 与圆 Γ 除了 M 以外的另一个交点为 D. 点 E 在线段 CB 的延长线上, 且 $2|EC| = |BC| + |CA| + |AB|$. 证明: $\angle EAB = \angle CAD$.

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第 5 题 设 n 是的大于 1 的整数. 已知 $a_1, a_2, \ldots, a_n$ 是 n 个两两不等的正奇数, 并且所有的差 $|a_i - a_j|$ 互不相等 $(1 \le i < j \le n)$. 证明:

$$\sum_{i=1}^{n} a_i \ge \frac{1}{2} n(n^2 - 4n + 5).$$

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第 6 题 设 $n>1$ 是给定的正整数. 若非负实数 $a_1, a_2, \ldots, a_n$ 满足

$$\frac{1}{k}\sum_{i=1}^{k} a_i \geq \frac{1}{k+1}\sum_{i=1}^{k+1} a_i, \qquad (k=1, 2, \ldots, n-1),$$

并且

$$(n^2-1)\left(\sum_{i=1}^{n} a_i\right)^3 = 3n^3.$$

证明: 一定存在正整数 $p \in \{2, 3, \ldots, n\}$, 使得

$$a_p^3 \leq \frac{1}{p^2-p}.$$

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Day 2 (October 6, 2018)

Problem 4. A triangle ABC is inscribed in circle Γ. Let M be the midpoint of the arc BC (arc BC does not contain the point A). Point J is the excenter of triangle ABC relative to B. The line JM meets circle Γ at two points M and D, Point E is on the extension of the line segment CB, such that $2|EC| = |BC| + |CA| + |AB|$. Prove that $\angle EAB = \angle CAD$.

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Problem 5. Let n be an integer greater than one. Suppose that a_1, a_2, ..., a_n are n distinct positive odd integers, whose differences $|a_i - a_j|$ (for $1 \leq i < j \leq n$) are also distinct. Show that

$$\sum_{i=1}^{n} a_i \geq \frac{1}{2}n(n^2 - 4n + 5).$$

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Problem 6. Let $n > 1$ be a given positive integer. Suppose that nonnegative real numbers a_1, a_2, ..., a_n satisfy

$$\frac{1}{k}\sum_{i=1}^{k} a_i \geq \frac{1}{k+1}\sum_{i=1}^{k+1} a_i, \qquad (k = 1, 2, \ldots, n-1),$$

and

$$(n^2 - 1)\left(\sum_{i=1}^{n} a_i\right)^3 = 3n^3.$$

Show that there exists a positive integer $p \in \{2, 3, \ldots, n\}$ such that

$$a_p^3 \leq \frac{1}{p^2 - p}.$$

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