第四届“刘徽杯”数学竞赛

第二天 (2021 年 11 月 14 日)

第 4 题 给定整数 $n \geq 3$. 若不全为零的实数 $a_1, a_2, \ldots, a_n$ 满足 $\sum_{i=1}^{n} a_i = 0$, 证明:

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^2\right)^7 \geq \frac{(n-1)^5}{(n-2)^2(n^2-n+1)^2(n^2-3n+3)^2}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^7\right)^2.$$

第 5 题 是否存在正整数 $n \geq 2$, 使得对任何正整数 m 以及任意满足下列 3 个条件的多重集 $\mathcal{A}$, 都存在 $\mathcal{A}$ 的非空真子集 $\mathcal{B}$ 使得 $\mathcal{B}$ 中所有元素之和为 m 的倍数?

- $\mathcal{A}$ 中互不相同的元素不超过 $n+2$ 个.
- $\mathcal{A}$ 中所有元素之和为 mn.
- 对任意的 $x \in \mathcal{A}$, x 是正整数且 $x \mid m$.

若存在, 请求出最小的满足上述条件的正整数 n; 否则请证明这样的 n 不存在.

注:

- 对多重集的元素求和时, 重复的元素需按照其重数累加.
- 多重集 $\mathcal{B}$ 是 $\mathcal{A}$ 子集是指对任意的 $x \in \mathcal{B}$ 都有 $x \in \mathcal{A}$, 并且 x 在 $\mathcal{B}$ 中的重数不超过 x 在 $\mathcal{A}$ 中的重数.

第 6 题 给定正整数 $t \geq 3$. 若无向图 G 有 $2t$ 个顶点, 且满足以下性质, 则称 G 为好图:

- 从 G 中任取 t 个顶点, 其中总存在两个顶点可以被 G 的边相连.
- 无法将 G 的顶点等分成两组, 使得这两组之间没有 G 的边相连.

求好图的边数的最小值.

定义一个图的分支序列为其各连通分支的顶点数目从小到大排列构成的序列. 求出使得边数达到最小的好图所有可能的分支序列.

Fourth Liu Hui Cup Mathematical Olympiad

Day 2 (November 14, 2021)

Problem 4. Let $n \geq 3$ be an integer, and $a_1, a_2, \ldots, a_n$ be real numbers not all zero such that $\sum_{i=1}^{n} a_i = 0$. Show that

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^2\right)^7 \geq \frac{(n-1)^5}{(n-2)^2(n^2-n+1)^2(n^2-3n+3)^2}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^7\right)^2.$$

Problem 5. Does there exist a positive integer $n \geq 2$ satisfying that for any positive integer m and any multiset $\mathcal{A}$ with the following properties, there exists a non-empty proper subset $\mathcal{B}$ of $\mathcal{A}$ such that the sum of all elements in $\mathcal{B}$ is a multiple of m?

- There are at most $n+2$ distinct elements in $\mathcal{A}$.
- The sum of all elements in $\mathcal{A}$ is mn.
- For any $x \in \mathcal{A}$, x is a positive integer and $x \mid m$.

If the answer is affirmative, find the smallest n; otherwise, prove that such an integer n cannot exist.
Remark:

- When taking the sum of elements in a multiset, repeated elements need to be accumulated according to the corresponding multiplicity.
- A multiset $\mathcal{B}$ is a subset of $\mathcal{A}$ if $x \in \mathcal{A}$ holds for any $x \in \mathcal{B}$, and, in addition, the multiplicity of x in $\mathcal{B}$ is less than or equal to the multiplicity of x in $\mathcal{A}$.

Problem 6. Given a positive integer $t \geq 3$. An undirected graph G with $2t$ vertices is called *good* if the following conditions are satisfied.

- Among any t vertices of G, there exist two vertices that are connected by an edge of G.
- It is impossible to separate G into two isolated parts with equal number of vertices. By isolated, we mean that there is no edge of G that connects these two parts.

Find the minimum number of edges of a good graph.
Define the *branch sequence* of a graph as the non-decreasing sequence formed by the number of vertices of each connected component. Find all possible branch sequences of a good graph whose number of edges is minimized.