THE HONG KONG POLYTECHNIC UNIVERSITY
香港理工大學

Secondary School Mathematics & Science Competition

Mathematics

Date: 1st May, 2013
Time allowed: 1 hour 15 minutes

1. Write your Name (both in English and Chinese), Name of School, Form, Date, Sex, Language, Subject and Candidate Number in the spaces provided on the *MC Answer Sheet* and the *Part B Answer Sheet*.
2. When told to open this question paper, you should check that all the questions are there. Look for the words '**END OF PAPER**' after the last question.
3. Answer **ALL** questions in Part A. You are advised to use an **HB** pencil to mark your answers on the MC Answer Sheet.
4. You should mark only **ONE** answer for each question in Part A. If you mark more than one answer, you will receive **NO MARKS** for that question.
5. Part B consists of Sections B(1), B(2) and B(3). Answer **ANY ONE** section. Answer any **FOUR** questions from your chosen section.
6. For Part B, answers should be an exact value or mathematical expressions unless otherwise specified.
7. No marks will be deducted for wrong answers.
8. The diagrams in the paper are not necessarily drawn to scale.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

PART A

Answer all questions.

Choose the best answer for each question.

1. If α and β are the roots of the quadratic equation $-5x(2x+1)=0$, find the value of $\alpha-\beta$.

A. $-\dfrac{1}{2}$

B. 0

C. $\dfrac{1}{2}$

D. $\dfrac{1}{2}$ or $-\dfrac{1}{2}$

2. Which of the following equations have two distinct real roots?

I. $x(x-1)=0$

II. $(x-1)(x-3)=(x-1)$

III. $x^2-1=0$

A. I and II only

B. I and III only

C. II and III only

D. I, II and III

3. Let $f(x)$ be a cubic polynomial with leading coefficient 2 and $f(1)=f(0.5)=f(-1)=-2$, then $f(x)=$

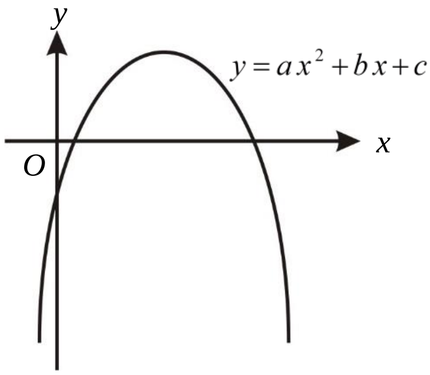
A. $(x+1)(x-1)(2x+1)-2$.

B. $(x+1)(x-1)(2x-1)-2$.

C. $(x+1)(x-1)(2x-1)+2$.

D. $(x+1)(x-1)(2x+1)+2$.

4. The figure shows the graph of $y = ax^2 + bx + c$. Which of the following is true?



A. $a < 0$, $c < 0$, $b^2 - 4ac < 0$
B. $a < 0$, $c < 0$, $b^2 - 4ac > 0$
C. $a > 0$, $c > 0$, $b^2 - 4ac < 0$
D. $a > 0$, $c > 0$, $b^2 - 4ac > 0$

5. Let $f(x)$ be a linear function in *x*.
It is known that $f(2013) \leq f(2014)$, $f(2015) \geq f(2016)$ and $f(2017) = 2018$.
Which of the following statements is true?

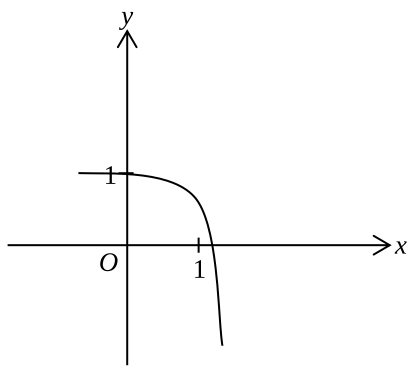
A. $f(2018) = 2018$
B. $f(2018) > 2018$
C. $f(2018) = 2019$
D. $f(2018) > 2019$

6. If log 2 = *a* and log 3 = *b*, then log 750 =

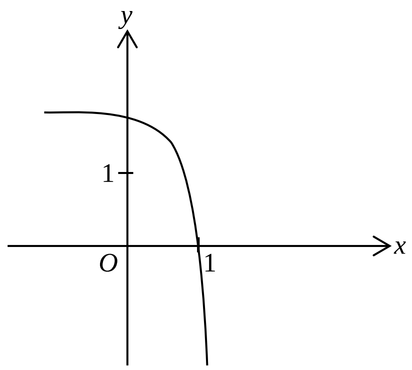
A. 3*a* – *b* + 2.
B. –3*a* + *b* + 2.
C. 2*a* – *b* + 3.
D. –2*a* + *b* + 3.

7. Which of the following may represent the graph of $y = a - a^x$, where $a > 0$ and $a \neq 1$?

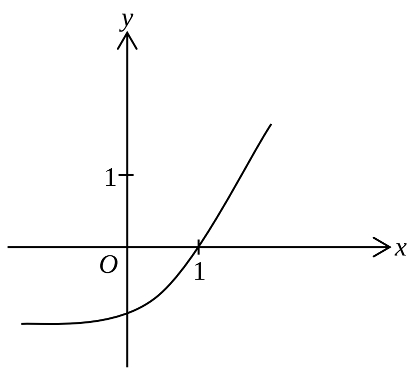
A.



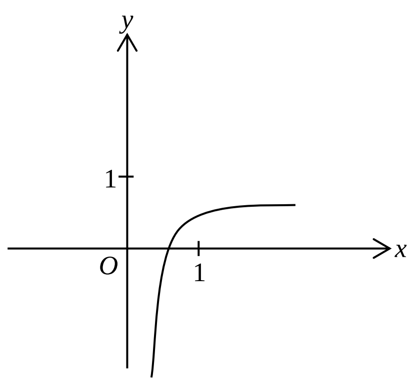
B.



C.



D.



8. If $\log_{2015}(\log_{2014}(\log_{2013} x)) = 1$, then $x =$

A. $2015^{2014^{2013}}$.

B. $2013^{2014^{2015}}$.

C. $2014^{2013^{2015}}$.

D. $2013^{2015^{2014}}$.

9. Find the remainder if the polynomial $x + x^2 + x^4 + x^8 + x^{16} + x^{32}$ is divided by $x + 1$.

A. 0
B. 2
C. 4
D. 6

10. Let $P(x)$ be a polynomial. If $x - k$ is a factor of $P(x)$, $x - k$ MUST also be a factor of which of the following polynomial(s)?

I. $2P(x)$

II. $[P(x)]^2$

III. $P(2x)$

A. I only
B. III only
C. I and II only
D. II and III only

11. Given that a and b are constants. If the polynomial ax^3+bx^2-5x+2 is divided by x^2-3x+2, the quotient is $Q(x)$ and the remainder is $12(x-1)$. Find $Q(x)$.

A. $x-1$
B. $x+1$
C. $2x-7$
D. $2x+7$

12. If a, b and c are rational numbers, which of the following MUST be correct?

I. If $ac=bc$, then $a=b$.
II. If $a^2=b^2$, then $a=b$.
III. If $a^3=b^3$, then $a=b$.

A. I only
B. II only
C. III only
D. II and III only

13. Two roots of the equation $x^3+mx^2+nx+2m=0$ are 4 and 7. The third root is

A. $\frac{11}{13}$.
B. $\frac{11}{15}$.
C. $\frac{11}{27}$.
D. $-\frac{11}{27}$.

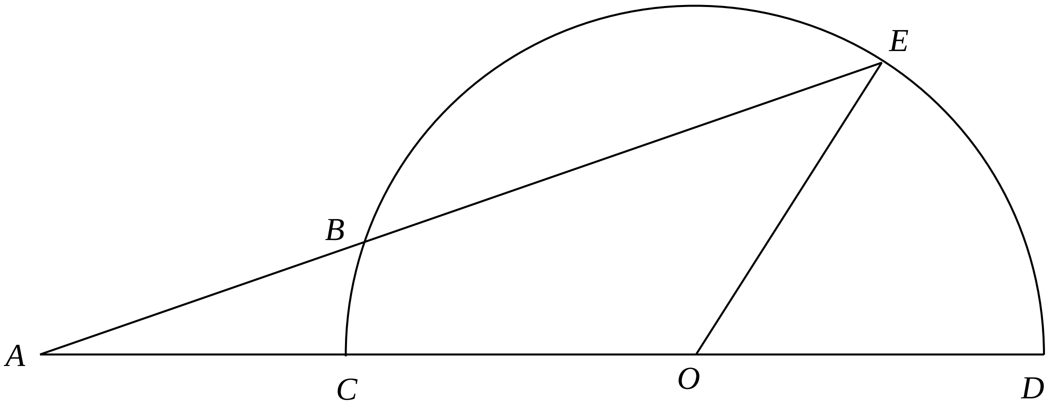
14. If $\dfrac{bc}{a}$ is a non-zero constant, then

A. a varies directly as b and c.
B. a varies inversely as b and inversely as c.
C. a varies directly as b and inversely as c.
D. a varies directly as c and inversely as b.

15. Let a, b, c and d be non-zero real numbers. If $a : c = b : c = d : b = 2 : 1$, then $d : a =$

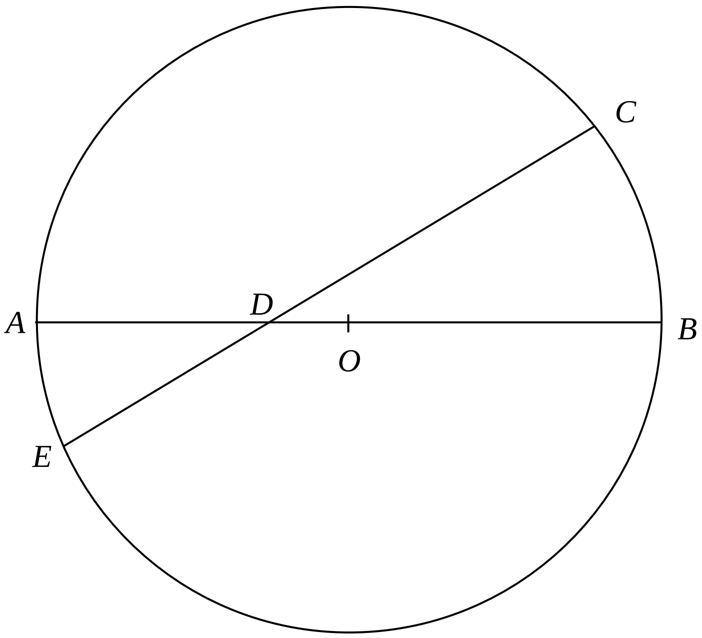
A. 1 : 1.
B. 2 : 1.
C. 4 : 1.
D. 8 : 1.

16. In the figure, a semi-circle with centre O is shown. ABE and ACD are straight lines. $\angle EOD = 45^\circ$. If $AB = OD$, find $\angle EAD$.



A. 15°
B. 17.5°
C. 20°
D. 25°

17. In the figure, a circle with centre O is shown. AB and CE intersect at D. It is given that $OD = 2$, $OB = 6$ and $\angle CDB = 30^\circ$. Find the length of CE.



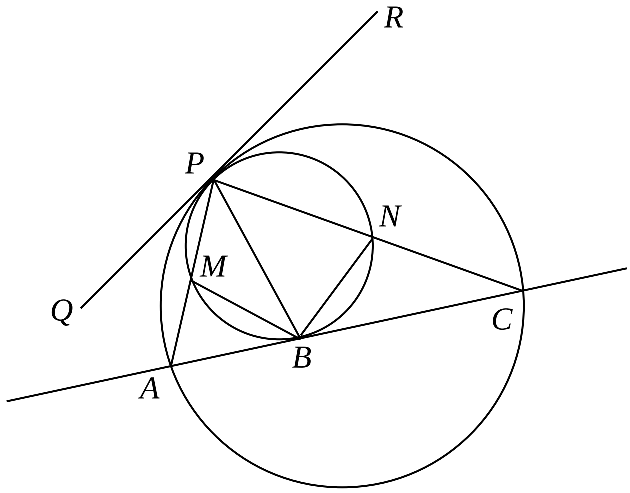
A. 12

B. $2\sqrt{35}$

C. $8\sqrt{2}$

D. 11

18. In the figure, two different circles touch internally at P. AC is tangent to circle $PMBN$ at B. QR is tangent to circle APC at P. Which of the following is/are correct?



I. $\angle APB = \angle CPB$

II. $\angle PAC = \angle PCA$

III. $\angle PMB = \angle PNB$

A. I only

B. III only

C. I and III only

D. II and III only

19. Let O be the origin in the rectangular coordinate plane. A point A is reflected about the y-axis to point B. If the slope of OA is m, then the slope of OB is

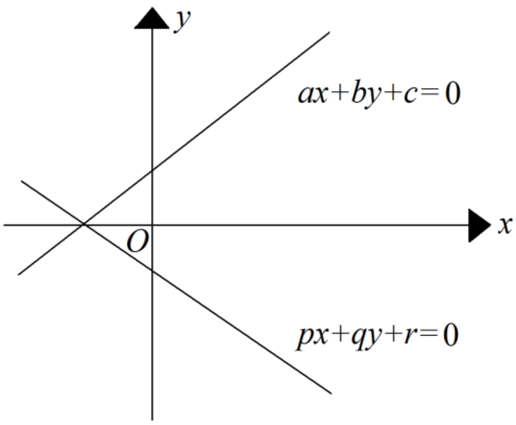
A. $-\dfrac{1}{m}$.

B. $\dfrac{1}{m}$.

C. $-m$.

D. m.

20. In the figure, two straight lines $ax+by+c=0$ and $px+qy+r=0$ intersect on the negative x-axis. Which of the following must be true?



I. $bc > 0$
II. $qr > 0$
III. $ar = cp$
IV. $br = -cq$

A. I and III only
B. I and IV only
C. II and III only
D. II and IV only

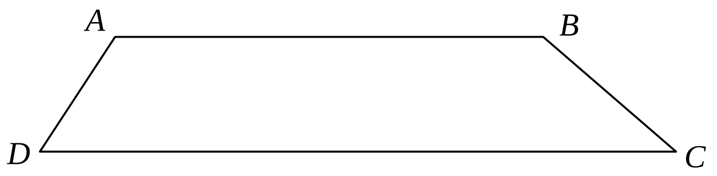
21. Find the acute angle between two straight lines $x+y+2=0$ and $\sqrt{3}x-y+1=0$.

A. 30°
B. 45°
C. 60°
D. 75°

22. There are m values of p and n values of q. It is given that $p > q$ and the mean of these $m + n$ values is greater than $\frac{p+q}{2}$. Find the median of these $m + n$ values.

A. p
B. q
C. $\frac{p+q}{2}$
D. Cannot be determined

23. Trapezium *ABCD* is shown in the figure, where $AB = 12.5$, $BC = 4$, $CD = 17.5$, $DA = 3$. Find the area of *ABCD*.

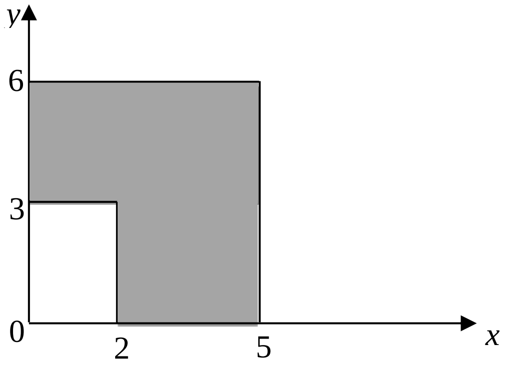


A. 33
B. 34
C. 35
D. 36

24. A circle is inscribed inside a rhombus. The lengths of diagonals of the rhombus are 30 and 40. Find the radius of the circle.

A. 10
B. 12
C. 15
D. 18

25. The figure shows a shaded region in the rectangular plane, with vertices at (2, 0), (5, 0), (5, 6), (0, 6), (0, 3), and (2, 3). A line $y = mx$ is to be drawn to divide the area of the shaded region in half. Find the value of *m*.



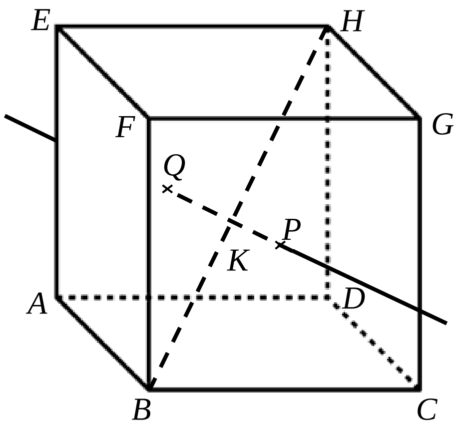
A. $\frac{3}{2}$

B. $\frac{6}{5}$

C. $\frac{7}{6}$

D. $\frac{8}{7}$

26. In the figure, *ABCDEFGH* is a unit cube. *K* is a moving point on the diagonal *BH* such that a straight line passing through *K* is perpendicular to *BH*. This straight line intersects the faces *BCGF* and *ADHE* at *P* and at *Q* respectively. Let $BK = x$ and $PQ = y$. Find the value of *x* when *y* is maximized.



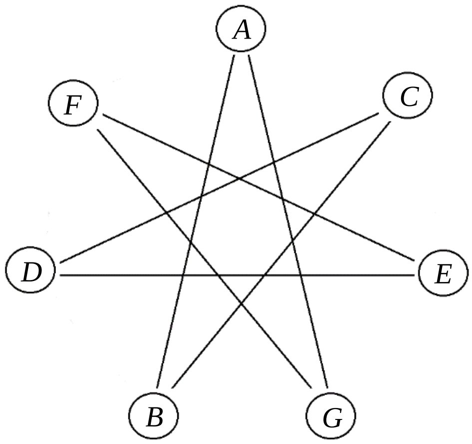
A. $\frac{\sqrt{2}}{2}$

B. $\frac{\sqrt{3}}{2}$

C. $\frac{1}{2}$

D. $\sqrt{2}$

27. In the figure, circles marked *A*, *B*, *C*, *D*, *E*, *F* and *G* are joined by line segments to form a star. Peter fills the numbers 7, 8, 9, 10, 11, 12 and 13 in the circles in a certain order. He calculates the sum of two numbers at the end of each line segment and find that the sums obtained from *AB*, *BC*, *CD*, *DE*, *EF*, *FG* and *GA* form an arithmetic sequence. Find the sum of this arithmetic sequence.



A. 35
B. 70
C. 140
D. 210

28. The graph of $y = x^4 + ax^3 + bx^2 + cx + d$, where *a*, *b*, *c*, *d* are real numbers, has four distinct *x*-intercepts. If one of the *x*-intercept is 0, which of the following must be non-zero?

A. a
B. b
C. c
D. d

29. Let n be a positive integer. If $\log_{18} 2 = a$, then $\log_3 2 + \log_9 4 + \cdots + \log_{3^n} 2^n =$

A. na.
B. $2na$.
C. $\dfrac{na}{1-a}$.
D. $\dfrac{2na}{1-a}$.

END OF PART A

PART B

Answer ANY ONE SECTION from Sections B(1), B(2) or B(3).

SECTION B(1)

Answer any FOUR questions.

31. [Arithmetic and Geometric Sequences and their Summations]

(a) Find the sum to infinity of geometric series $4+\frac{4}{\sqrt{2}+1}+\frac{4}{(\sqrt{2}+1)^2}+\frac{4}{(\sqrt{2}+1)^3}+\cdots$.

Express the answer in the form $a+b\sqrt{2}$, where a and b are integers.

(2 marks)

(b) If the geometric series $4+\frac{4}{\sqrt{2}-k}+\frac{4}{(\sqrt{2}-k)^2}+\frac{4}{(\sqrt{2}-k)^3}+\cdots$ can be summed to infinity, find the range of values of k.

(2 marks)

32. [Arithmetic and Geometric Sequences and their Summations]

Consider an arithmetic sequence

$$a_1, a_2, a_3, \cdots, a_8,$$

where all terms are distinct. The terms a_1, a_5 and a_8 form the first three terms of a geometric sequence, and $a_1+a_5+a_8=148$. Let d be the common difference of the arithmetic sequence.

(a) Express a_1 in terms of d.

(1 marks)

(b) Find the 5^{th} term of the geometric sequence.

(3 marks)

33. [Equations of Circles]

A circle touches both x-axis and y-axis, and passes through the point (4, 2). Find the two possible equations of the circle.

(4 marks)

34. [Locus]

A point P on the rectangular plane is equidistant from the straight line $L : y = 1$ and a fixed point A (2, –1).

(a) Find the equation of locus of P.

(2 marks)

(b) Hence, find the range of values of y in the above equation.

(2 marks)

35. [Probability, Permutation and Combination]

(a) A fair die is rolled six times. Find the probability that each of the six different numbers will appear exactly once. (Express the answer correct to 3 significant figures.)

(2 marks)

(b) A fair die is rolled seven times. Find the probability that each of the six different numbers will appear at least once. (Express the answer correct to 3 significant figures.)

(2 marks)

36. [Probability, Permutation and Combination]

A large shipment of tablets are divided into batches. Each batch is subject to inspection by the following double sampling inspection plan:

I. A random sample of 10 tablets is drawn from a batch and inspected. If none of the tablets is defective, the batch is accepted.

II. If 2 or more tablets are defective, the batch is rejected. If 1 tablet is defective, another sample of 10 tablets is drawn from the batch and inspected. If none of the tablets from the 2^{nd} sample is defective, the batch is accepted; otherwise it is rejected.

It is known that the shipment contains 5% defective tablets. Find the probability that

(a) a batch is accepted. (Express the answer correct to 4 significant figures.)

(2 marks)

(b) the 4^{th} batch is the first to be rejected if the tablets are inspected batch by batch.

(Express the answer correct to 4 significant figures.)

(2 marks)

END OF SECTION B(1)

SECTION B(2)

Answer any FOUR questions.

37. [Binomial Expansion, Exponential and Logarithmic Functions]

Let n be a positive integer. If the expansion of $\left(x^3+\dfrac{1}{x^2}\right)^n$ has a constant term, find all possible values of n.

(4 marks)

38. [Binomial Expansion, Exponential and Logarithmic functions]

Solve the following equations:

(a) $e^{2x}+3e^x-4=0$

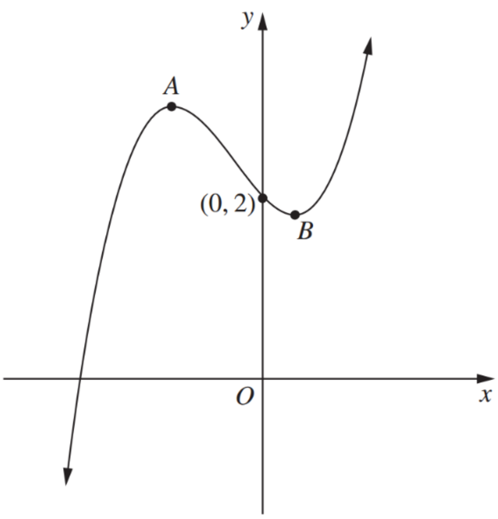
(2 marks)

(b) $2\log_3 x=\log_3 2x-2$

(2 marks)

39. [Differentiation]

The graph of $y=x^3+\dfrac{5}{2}x^2-2x+2$ is shown below. Points A and B are turning points.



(a) Find the coordinates of A and B.

(2 marks)

(b) The equation $2x^3+5x^2-4x=k$ has three distinct real roots. Find the range of values of k.

(2 marks)

40. [Differentiation]

A solid circular cylinder with height h cm and radius r cm has a total surface area of 150π cm^2. Its volume is V cm^3.

(a) Express V in terms of r.

(2 marks)

(b) Find the maximum possible value of V. (Express the answer in terms of π .)

(2 marks)

41. [Integration]

Suppose $\frac{d}{dx}F(x)=\frac{(\ln x)^5}{x}$. If $F(1)=0.2013$, find the value of $F(2)$. (Express the answer correct to 4 decimal places.)

(4 marks)

42. [Integration]

Let $\frac{x^2+1}{x(x-1)^2}=\frac{A}{x}+\frac{B}{(x-1)^2}$.

(a) Find the values of A and B.

(2 marks)

(b) Evaluate. $\int_2^3 \frac{x^2+1}{x(x-1)^2}\,dx$

(2 marks)

END OF SECTION B(2)

SECTION B(3)

Answer any FOUR questions in this section.

43. [Binomial expansion, Exponential and logarithmic functions]

Let n be a positive integer. If the expansion of $\left(x^3 + \dfrac{1}{x^2}\right)^n$ has a constant term, find all possible values of n.

(4 marks)

44. [Matrix]

Consider the following system of homogenous linear equations in x, y and z:

$$\begin{cases} x + ky - 5z = 0 \\ 3x + y + z = 0 \\ 5x + y + kz = 0 \end{cases},$$

where k is a real number. If the system has non-trivial solutions, find the value(s) of k.

(4 marks)

45. [Differentiation]

A solid circular cylinder with height h cm and radius r cm has a total surface area of 150π cm^2. Its volume is V cm^3.

(a) Express V in terms of r.

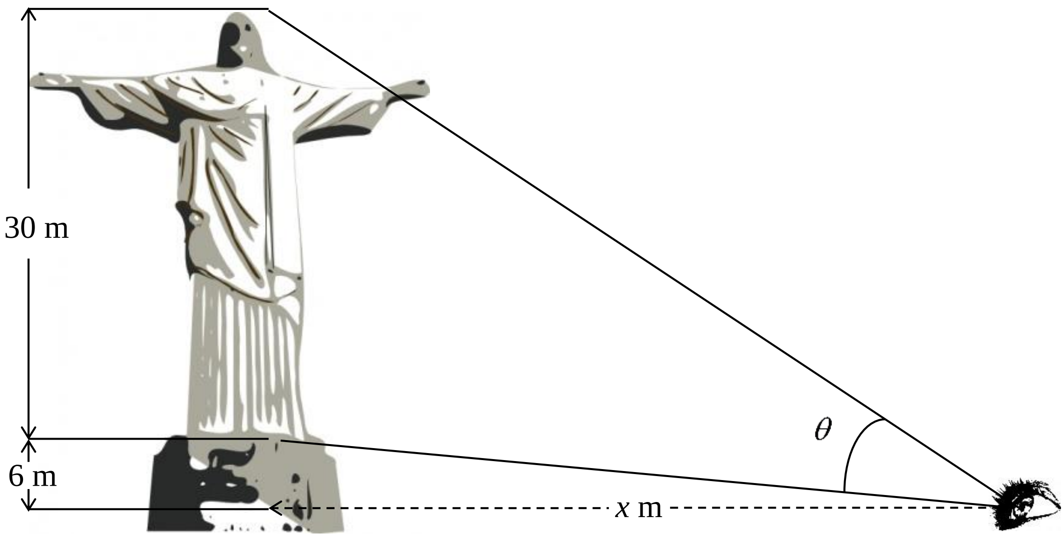
(2 marks)

(b) Find the maximum possible value of V. (Express the answer in terms of π.)

(2 marks)

46. [Differentiation]

A statue of 30 m high stands on a pedestal. Nelson stands at x m from the base of the pedestal and looks at the statue. He wants to find out how far away he should stand from the base of the pedestal to make the statue appear the largest, i.e., the angle θ subtended by his lines of sight to the bottom and the top of the statue is the largest.



It is given that Nelson's eye-level is 6 m below the bottom of the statue.

(a) Express $\tan\theta$ in terms of x.

(2 marks)

(b) Find the value of x when $\tan\theta$ is maximized. (Express the answer in surd form.)

(2 marks)

47. [Integration]

Let $F(x)$ be an anti-derivative of $\frac{e^x}{x}$, where $x > 0$. If $\int_2^7 \frac{e^{x^2}}{x}\,dx = c\left(F(b) - F(a)\right)$, find the values of a, b, and c.

(4 marks)

48. [Integration]

Let $\dfrac{x^2+1}{x(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{(x-1)^2}$.

(a) Find the values of *A* and *B*.

(2 marks)

(b) Evaluate $\int_2^3 \dfrac{x^2+1}{x(x-1)^2}\,dx$.

(2 marks)

END OF SECTION B(3)

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