THE HONG KONG POLYTECHNIC UNIVERSITY
香港理工大學

Secondary School Mathematics & Science Competition 2015

Mathematics

Date	: 1 May 2015	Total no. of pages	: 18
Time allowed	: 9:30 a.m. - 10:45 a.m. (1 hour 15 minutes)	Total marks	: 76

1. Write your Candidate Number, Exam Centre Number, Seat Number, Name in English, Name of School, Form, Language and Subject in the spaces provided on the Part A MC Answer Sheet and the Part B Answer Sheet.

2. When told to open this question paper, you should check that all the questions are there. Look for the words '**END OF PAPER**' after the last question.

3. Answer **ALL** questions in Part A (60 marks).

   (a) You are advised to use an **HB** pencil to mark all your answers on the MC Answer Sheet.

   (b) Each question carries **TWO marks**.

   (c) You should mark only **ONE** answer for each question. If you mark more than one answer, you will receive **NO MARK** for that question.

4. Part B (16 marks) consists of Sections B1, B2 and B3.

   (a) Answer **ANY ONE SECTION**. Answer **ANY FOUR** questions from your chosen section.

   (b) Unless otherwise stated, your answers may be exact values or mathematical expressions.

   (c) Answers should be written in the space provided on the Part B Answer Sheet.

5. No mark will be deducted for wrong answers.

6. The diagrams in this paper are not necessarily drawn to scale.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

Part A: Multiple Choice Questions (60 marks) (Each question carries **TWO** marks)

1. It is given that the equation $3x^2 - 5x + (k-1) = 0$ has real roots, where k is a positive integer. Find the number of possible value(s) of k.

   A. 1
   B. 2
   C. 3
   D. 4

2. If α and β are the roots of the equation $x^2 + ax + b = 0$, which of the following quadratic equation in x with roots $\alpha - 1$ and $\beta - 1$?

   A. $x^2 - (a-1)x + (b-1) = 0$

   B. $x^2 + (a-1)x + (b-1) = 0$

   C. $x^2 - (a+2)x + (a+b+1) = 0$

   D. $x^2 + (a+2)x + (a+b+1) = 0$

3. If $g(x) = \dfrac{x}{2x+1}$, then $g\left(\dfrac{2}{x}\right) \cdot g\left(\dfrac{x}{2}\right) =$

   A. 1.
   B. 2.
   C. $\dfrac{x(x+1)}{4}$.
   D. $\dfrac{x}{(x+1)(x+4)}$.

4. Let $f(x) = ax^2 + bx + c$, where a, b and c are real numbers and $a : b = b : c$. If (h, k) is the vertex of the graph of $y = f(x)$, which of the following must be true?

I. $ck > 0$
II. $abh < 0$
III. $c > k$

A. I only
B. III only
C. I and II only
D. II and III only

5. Let $y = -3 - 2(1 - x)^2$. Which of the following about its graph is/are true?

I. The coordinates of the vertex are $(1, -3)$.
II. The y-intercept is -3.
III. The line of symmetry is $x = -1$.

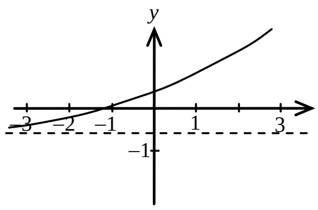
A. I only
B. II only
C. I and III only
D. II and III only

6. Let $a > 0$ and $b > 0$. If $\log_8 a = \log_2 b$, then $\log a : \log b =$

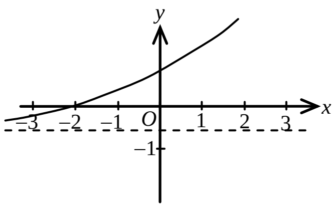
A. 1 : 4.
B. 4 : 1.
C. 1 : 3.
D. 3 : 1.

7. Which of the following represents the graph of $y=\left(\dfrac{3}{4}\right)^{x+1}-1$?

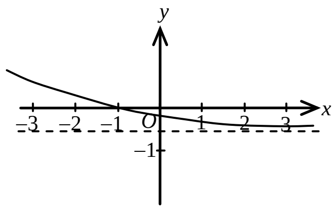
A.



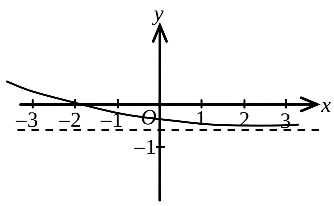
B.



C.



D.



8. Simplify $\dfrac{2^{-2}\log x+\log\sqrt{y}}{2\log x+\log y^{4}}$.

A. $\dfrac{1}{6}$

B. $\dfrac{1}{7}$

C. $\dfrac{1}{8}$

D. $\dfrac{1}{9}$

9. Let p and q be non-zero real numbers. When $x^{2015} + px^{2013} + qx^{2011} + 2x$ is divided by $x - 2015$, the remainder is 1. Find the remainder when $x^{2015} + px^{2013} + qx^{2011} + 2x$ is divided by $x + 2015$.

A. –1
B. 1
C. 2
D. 2015

10. It is given that the LCM and HCF of two polynomials are $2x\left(4x^2 - 1\right)\left(x^2 - 1\right)$ and $x - 1$ respectively. If one of the polynomials is $2x^3 - x^2 - 2x + 1$, find the other polynomial.

A. $2x^3 - 2x^2 - 2x$
B. $2x^3 - x^2 - x$
C. $4x^3 + 2x^2 - 2x$
D. $4x^3 - 2x^2 - 2x$

11. If $x^3 - 3x^2 + ax - 21$ is divisible by $x^2 + bx + 3$, then b =

A. –4.
B. –2.
C. 2.
D. 4.

12. Solve the equation $\dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{x}}} = x$.

A. $\sqrt{2}$ or $-\sqrt{2}$
B. $\dfrac{1}{\sqrt{2}}$ or $-\dfrac{1}{\sqrt{2}}$
C. $\dfrac{1}{2}$ or $-\dfrac{1}{2}$
D. $\dfrac{1}{2}$

13. Suppose $0^\circ \le \theta \le 360^\circ$. How many roots does the equation

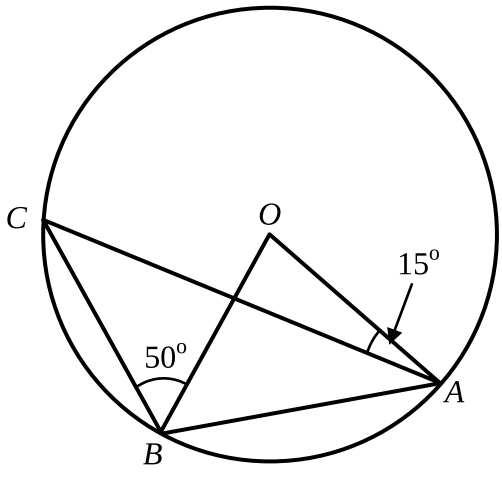
$\sin^2(180^\circ + \theta) + 7\cos(270^\circ - \theta) = 4$ have?

A. 0
B. 1
C. 2
D. 4

14. It is given that x varies directly as y^2 and inversely as $\sqrt{z}$. Which of the following expressions must be a constant?

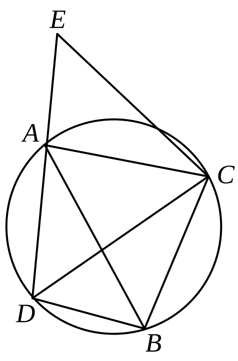
A. $\dfrac{xy^2}{\sqrt{z}}$
B. $\dfrac{y^2\sqrt{z}}{x}$
C. $\dfrac{y^4}{x^2 z}$
D. $\dfrac{x^2 y^4}{z}$

15. In the figure, O is the centre of the circle, $\angle OAC = 15^\circ$ and $\angle OBC = 50^\circ$. Find $\angle CAB$.



A. 35°
B. 40°
C. 50°
D. 55°

16. In the figure, $\triangle ABC$ is inscribed in a circle where $AC = BC$. $\triangle CDE$ is another isosceles triangle such that $CD = CE$ and DE passes through A. If $AC \perp BC$, then $AD + BD = kCD$. Find the value of k.



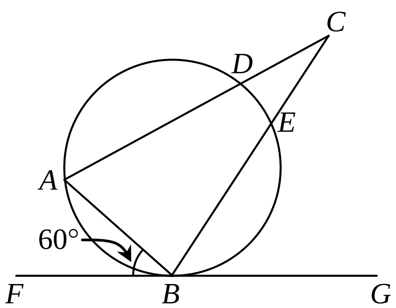
A. 1

B. $\sqrt{2}$

C. $\sqrt{3}$

D. 2

17. In the figure, $ABED$ is a circle. FG is a straight line tangent to the circle at B. $\angle ABF = 60^\circ$ and $\overset{\frown}{AB} : \overset{\frown}{DE} = 3:1$. Find $\angle ACB$.



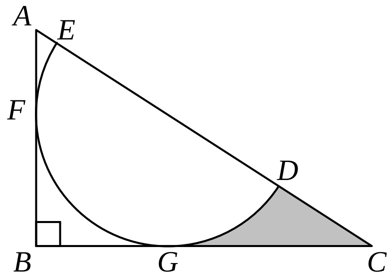
A. 20°

B. 30°

C. 35°

D. 40°

18. In the figure, $AB \perp BC$, $AB = 6$ and $BC = 8$. $DEFG$ is a semicircle with diameter DE and touches AB and BC at F and G respectively. Find the area of the shaded region correct to 2 decimal places.



A. 2.35
B. 2.37
C. 2.39
D. 2.41

19. Let $A(0,1)$, $B(2,-3)$ and $C(-4,-1)$ be the vertices of $\triangle ABC$. Find the coordinates of the orthocentre of $\triangle ABC$.

A. $\left(-\frac{2}{3},-1\right)$

B. $\left(-\frac{3}{5},-\frac{3}{4}\right)$

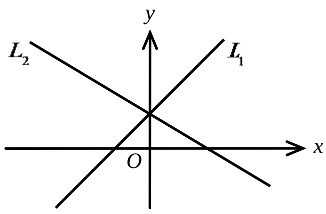
C. $(-1,-2)$

D. $(0,1)$

20. A sequence $a_1, a_2, a_3, \ldots$ is defined as $a_1 = a_2 = 2$ and $a_{n+2} = a_n a_{n+1}$, where n is a positive integer. The first few terms of the sequence are $2, 2, 4, 8, 32, \cdots$. Find the unit digit of a_{2015}.

A. 2
B. 4
C. 6
D. 8

21. In the figure, the equations of straight lines L_1 and L_2 are $px - y + b = 0$ and $x + py - c = 0$ respectively, where $p, b, c > 0$. L_1 and L_2 intersect at $(0, b)$. Which of the following must be true?



I. $L_1 \perp L_2$

II. $c = pb$

III. $b < -cp$

A. I only
B. II only
C. I and II only
D. I, II and III

22. A straight line L cuts the x-axis and y-axis at A and B respectively. If the centroid of $\triangle OAB$ is $(4, 6)$, where O is the origin, find the equation of L.

A. $3x + 2y - 36 = 0$
B. $3x + 2y - 18 = 0$
C. $3x - 2y = 0$
D. $3x + 2y = 0$

23. It is given that $a \neq 0$ and $d \neq 0$. Find the variance of the following numbers:

$$a - 2d, \; a - d, \; a, \; a + d, \; a + 2d.$$

A. 0
B. d
C. d^2
D. $2d^2$

24. Two sets of real numbers are given as follows:

Set A: α, β, γ, δ, θ

Set B: α, β, γ, δ, θ, μ

It is given that μ is the mean of numbers in Set A.

Which of the following must be true?

I. The mean of numbers in Set A is the same as that of numbers in Set B.

II. The inter-quartile range of numbers in Set A is the same as that of numbers in Set B.

III. The variance of numbers in Set A is the same as that of numbers in Set B.

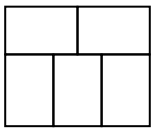
A. I only

B. I and II only

C. I and III only

D. I, II and III

25. The rectangle in the figure is sub-divided into five identical small rectangles. The perimeter of each of the small rectangles is 20. Find the total area of the five small rectangles.



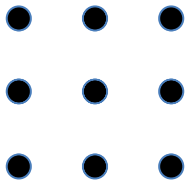
A. 80

B. 105

C. 120

D. 125

26. In the figure, nine grid points are shown. Triangles with all vertices lying on grid points are drawn. How many such triangles can be drawn?



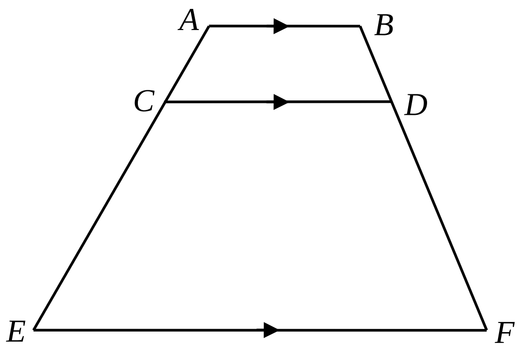
A. 72

B. 76

C. 80

D. 84

27. In the figure, ACE and BDF are straight lines, AB // CD // EF, $AB : CD : EF = 2 : 3 : 6$. Let y be the area of quadrilateral $ABDC$ and let z be the area of quadrilateral $CDFE$. Find $y : z$.



A. 1 : 2
B. 1 : 4
C. 5 : 9
D. 5 : 27

28. Let x be the number of digits of 2^n and let y be the number of digits of 2^{2014} under the decimal system, where n is a positive integer. It is given that $x > y$. Find the minimum value of n.

A. 2015
B. 2016
C. 2017
D. 2018

29. Let $M = \underbrace{111\cdots111}_{2015\text{ digits}}$. Find the sum of digits of $2015 \times M$.

A. 16120
B. 16112
C. 16104
D. 16096

30. Which of the following equations has the sum of roots (real or complex) equal to 4 and the product of roots equal to 2?

A. $x^3 - 4x^2 + 6x - 2 = 0$
B. $x^3 + 4x^2 + 6x - 2 = 0$
C. $x^3 + 6x^2 + 4x - 2 = 0$
D. $x^3 - 2x^2 + 6x - 4 = 0$

Part B (16 marks) (Answer <u>ANY ONE Section</u>)

Section B1 (Answer ANY FOUR questions)

31. (a) The p-th term and the q-th term of an arithmetic sequence are $\frac{1}{q}$ and $\frac{1}{p}$ respectively. Find the sum of the first pq terms of the sequence. (Express the answer in terms of p and q.)

(2 marks)

(b) The sum to infinity of a geometric sequence is 16. The sum to infinity of the squares of its terms is $\frac{768}{5}$. Find the sum of the first 4 terms of the sequence.

(2 marks)

32. Jacky borrowed \$3000 from a lending company with interest at 24% p.a. compounded monthly. Jacky is only able to meet the monthly minimum repayment (the greater of (opening balance + interest) × 5%, or \$50).

(a) Find the values of x and y in the following table:

n^{th} bill	Opening balance (\$)	Interest (\$)	Minimum payment (\$)
$n = 1$	3000	60	153
$n = 2$	2907	x	y

(2 marks)

(b) From the m^{th} bill and onwards, the minimum repayment is \$50. Find the value of m.

(2 marks)

33. Two circles $D_1 : x^2 + y^2 - 10x - 4y + 21 = 0$ and $D_2 : x^2 + y^2 - 20x - 14y + 131 = 0$ touch externally at point P.

(a) Find the coordinates of P.

(2 marks)

(b) Find the equation of the common tangent to D_1 and D_2 at P. Express the answer in the form $Ax + By + C = 0$, where A, B and C are real numbers.

(2 marks)

34. A straight line passing through the origin with slope m cuts the circle $x^2 + y^2 - 4x - 2y - 4 = 0$ at points A and B.

(a) Let P be the midpoint of AB. Find the coordinates of P in terms of m.

(2 marks)

(b) Hence, find the equation of the locus of P as m varies.

(2 marks)

35. An integer is randomly selected from 10000 to 99999 (including 10000 and 99999). Find the probabilities that

(a) the sum of all digits of the integer is 3;

(2 marks)

(b) all digits of the integer are distinct.

(2 marks)

36. A game is designed with 7 Stages. In each Stage, there are 5 multiple choice questions, and each question has 5 options. In order to win a Stage, more than 3 questions must be answered correctly. To win the game, 4 Stages must be won.

(a) If Bob plays the game by randomly guessing the answers to all the questions, find the probability that he wins a Stage. (Give your answer correct to 4 decimal places)

(2 marks)

(b) Suppose Peter won Stage 1 and Stage 2, and the probability that he answers a question correctly is 0.9. Find the probability that Peter wins the game. (Give your answer correct to 4 decimal places)

(2 marks)

END OF SECTION B1

Section B2 (Answer ANY FOUR questions)

37. (a) The sum of coefficients of the first three terms of the expansion in descending powers of $\left(x-\frac{3}{x^2}\right)^m$ is 559, where m is a positive integer. Find the coefficient of the fifth term of the expansion. (2 marks)

(b) It is given that $\frac{(1+2x)^n}{e^{ax}}=1-14x^2+$ terms involving higher powers of x, where a and n are positive integers. Find the value of n. (2 marks)

38. **Figure B2-1** shows the graph of $C: y=e^{-x}$. P is a moving point on C and L is the tangent to C at P. L cuts the x-axis and y-axis at $A(a, 0)$ and $B(0, b)$ respectively.

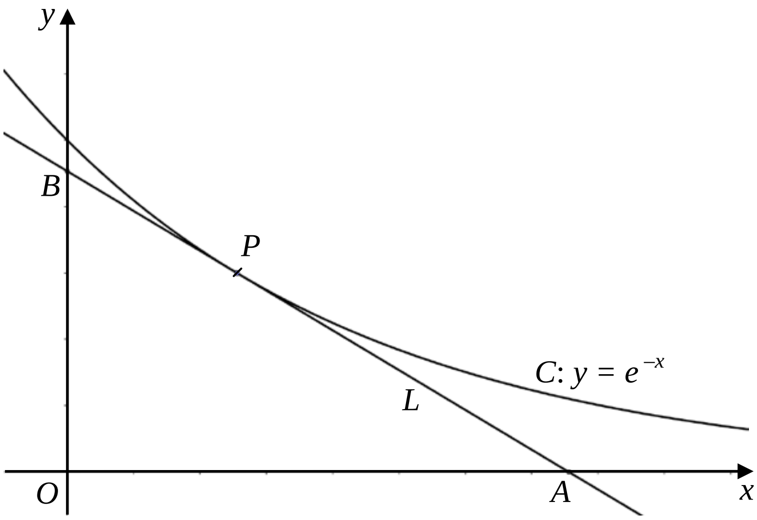


Figure B2-1

(a) Express b in terms of a. (2 marks)

(b) If A moves away from O at a uniform speed of 4 units per second, find the speed of B moving towards O when $OA=2$. (2 marks)

39. Let $f(x)=\frac{\log x}{x}$, where $x>0$.

(a) Find the maximum value of $f(x)$. (2 marks)

(b) Find the range of x such that the graph of $y=f(x)$ is concave upwards. (2 marks)

40. In **Figure B2-2**, the shaded region is bounded by the curves $y = x^2$ and $y = kx^3$, where $k > 0$. The two curves meet at *A*.

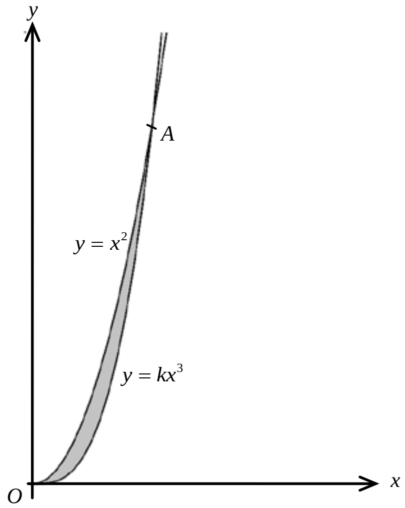


Figure B2-2

(a) Find the coordinates of A in terms of k. (2 marks)

(b) If the area of the shaded region is $\frac{9}{4}$, find the value of k. (2 marks)

41. An energy saving device is installed in a company. The device is expected to reduce the company's electricity bill at a rate of $s(t) = 150e^{0.5t}$ hundred dollars per year, where *t* is the number of years since the system is installed in 2015.

(a) Find the total reduction in the company's electricity bill during the first five years. (Give the answer correct to the nearest hundred dollars.) (2 marks)

(b) If the system costs \$600 000, how many years after 2015 will the reduction in electricity bill cover the cost? (Give the answer correct to the nearest integer.) (2 marks)

42. Let $f'(x) = \dfrac{(x-2)\left(\ln\left(x^2-4x+5\right)\right)^4}{x^2-4x+5}$. It is given that the minimum value of $f(x)$ is 0.

(a) Find $f(4)$. (2 marks)

(b) Let *A* be the area bounded by the curve $y = f(x)$, the *x*-axis, the *y*-axis and the line $x = 4$. Using the trapezoidal rule with 4 subintervals, find an estimate for *A*. Correct your answer to 4 decimal places. (2 marks)

END OF SECTION B2

Section B3 (Answer ANY FOUR questions)

43. (a) The sum of coefficients of the first three terms of the expansion in descending powers of $\left(x-\dfrac{3}{x^2}\right)^m$ is 559, where m is a positive integer. Find the coefficient of the fifth term of the expansion. (2 marks)

(b) It is given that $\dfrac{(1+2x)^n}{e^{ax}}=1-14x^2+$ terms involving higher powers of x, where a and n are positive integers. Find the value of n. (2 marks)

44. 1, 1, 2, 3, 5, 8, 13, … is known as the Fibonacci sequence. It is known that the matrix $Q=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$ is related to this sequence.

(a) Find a positive integer n such that $Q^n=\begin{pmatrix}13 & 8\\ 8 & 5\end{pmatrix}$. (2 marks)

(b) Evaluate $\det\left(Q^{2015}\right)$. (2 marks)

45. Let $f\left(x\right)=\dfrac{\log x}{x}$, where $x>0$.

(a) Find the maximum value of $f\left(x\right)$.

(2 marks)

(b) Find the range of x such that the graph of $y=f(x)$ is concave upwards.

(2 marks)

46. **Figure B3-1** shows a sector with subtended angle θ cut away from a circle of radius 4 units. The remaining part (i.e., the shaded part) is then folded to form a cone, where $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$.

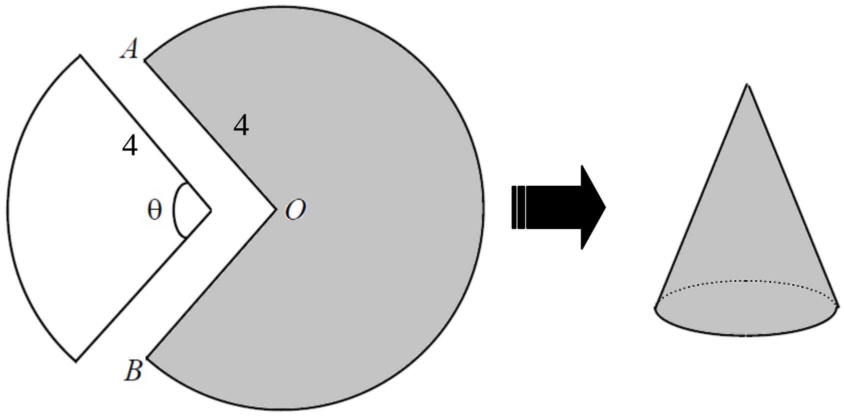


Figure B3-1

(a) Let S be the total surface area of the cone. If $S = \frac{4}{\pi}\left(\theta^2 - p\theta + q\right)$, find the values of p and q. Express your answers in terms of π.

(2 marks)

(b) Find the maximum value of S in terms of π.

(2 marks)

47. Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta$, where n is a positive integer and $n \ge 2$.

(a) If $I_n = kI_{n-2}$, find k in terms of n.

(2 marks)

(b) Evaluate I_5.

(2 marks)

48. (a) In **Figure B3-2**, part of function $y = -x^2 + 6$ is shown. Find the volume, in terms of h, of the solid obtained by revolving the shaded area about the y-axis.

(2 marks)

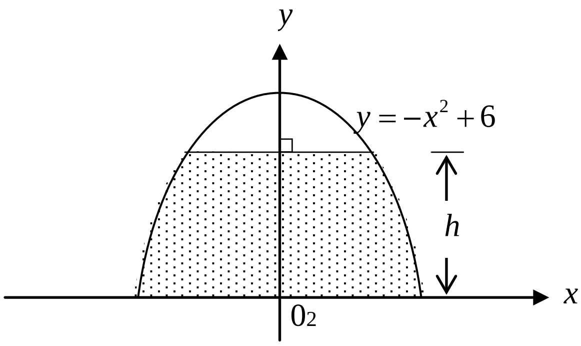


Figure B3-2

(b) In **Figure B3-3**, the solid obtained in (a) is placed in a cylindrical vessel of base radius 3 units and height 10 units. Let h be the depth of water.

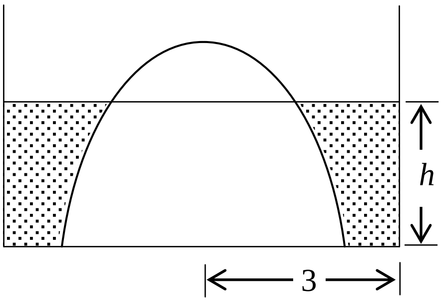


Figure B3-3

Water is being poured into the vessel at a rate of 10π cubic units per second. Find the rate of change of the depth of water when $h = 2$.

(2 marks)

END OF SECTION B3

END OF PAPER