THE HONG KONG POLYTECHNIC UNIVERSITY
香港理工大學

Secondary School Mathematics & Science Competition

Mathematics

5th May, 2012

1 hour 15 minutes

1. Write your Student Number, English Name, Subject and Date in the spaces provided on the "MC Answer Sheet".
2. Write your Student Number and English Name in the spaces provided on the Part B "Fill In The Blanks Answer Sheet".
3. When told to open this question paper, you should check that all the questions are there. Look for the words '**END OF PAPER**' after the last question.
4. **ANSWER ALL QUESTIONS** in Part A. You are advised to use an **HB** pencil to mark your answers on the MC Answer Sheet.
5. You should mark only **ONE** answer for each question in Part A. If you mark more than one answer, you will receive **NO MARKS** for that question.
6. Part B consists of Section B1, B2 and B3. **ANSWER EITHER** Section B1, B2 **OR** B3. **ANSWER 5 QUESTIONS** from your chosen Section **ONLY**.
7. For Part B, answers may be an exact value or mathematical expressions.
8. **NO MARKS** will be deducted for wrong answers in Part A and Part B.
9. The diagrams in the paper are not necessarily drawn to scale.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

PART A

ANSWER ALL questions in this part

Choose the best answer for each question.

1. $4^{ab} =$

   A. $2^a \times 2^b$.
   B. $4^a \times 4^b$.
   C. $4^a + 4^b$.
   D. $(2^b)^{2a}$.

2. Factorize $6ab - 9a^2 - b^2$.

   A. $-(3a-b)^2$
   B. $(3a-b)^2$
   C. $(-3a-b)^2$
   D. Cannot be factorized

3. If $x = \dfrac{2}{1+z} - y$, then $z =$

   A. $z = \dfrac{1}{x+y}$.
   B. $z = \dfrac{2-x-y}{x}$.
   C. $z = \dfrac{2-x+y}{x+y}$.
   D. $z = \dfrac{2-x-y}{x+y}$.

4. Which of the following statement(s) is/are true?
   I. $1-x^3 \equiv (1-x)(1+x+x^2)$
   II. $1-x^3 \equiv (1-x)^3$
   III. $1-x^3 \equiv (1-x)\left(\dfrac{-1+\sqrt{3}\,i}{2} - x\right)\left(\dfrac{-1-\sqrt{3}\,i}{2} - x\right)$

   A. I only
   B. II only
   C. I and III only
   D. II and III only

5. If $(x , y) = (2 , 1)$ is the solution of $\begin{cases} ax+by=17 \\ bx+cy=9 \end{cases}$, which of the following is correct?

A. $4a+c=25$
B. $2a+c=25$
C. $4a-c=25$
D. $2a-c=25$

6. Find the least integral value of k such that the equation $x^2-4x+k=0$ has no real roots.

A. 5
B. 4
C. -5
D. -4

7. If the roots of the equation $(a-2)x^2+(a^2-4)x-1=0$ are equal in magnitude but opposite in sign, then $a =$

A. ± 2.
B. ± 4.
C. -2.
D. 2.

8. If $f(x)=\dfrac{x+1}{x-1}$, then $f\left(\dfrac{1}{x}\right)$ =

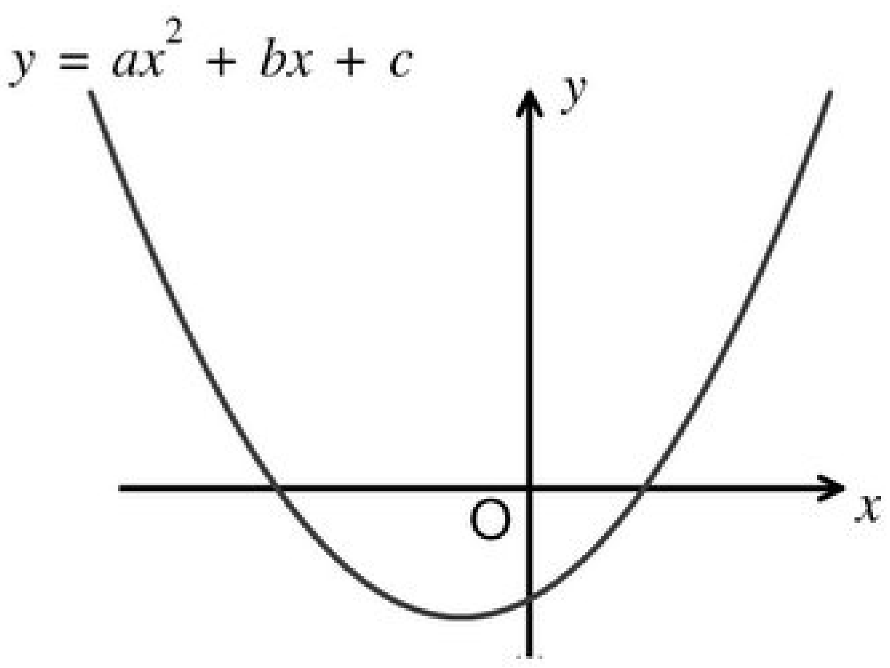
A. $f(x)$.
B. $-f(x)$.
C. $f\left(\dfrac{1}{x}\right)$.
D. $-f\left(\dfrac{1}{x}\right)$.

9. Let $f(x)$ be a function such that $f(x-1)=2x^2-x+7$.

Which of the following is $f(x+2)$?

A. $2x^2-x+10$
B. $2x^2+3x+8$
C. $2x^2+7x+13$
D. $2x^2+11x+22$

10. The figure shows the graph of $y=ax^2+bx+c$, where a, b and c are constants. Which of the following is/are true?



I. $a>0$
II. $b>0$
III. $c<0$

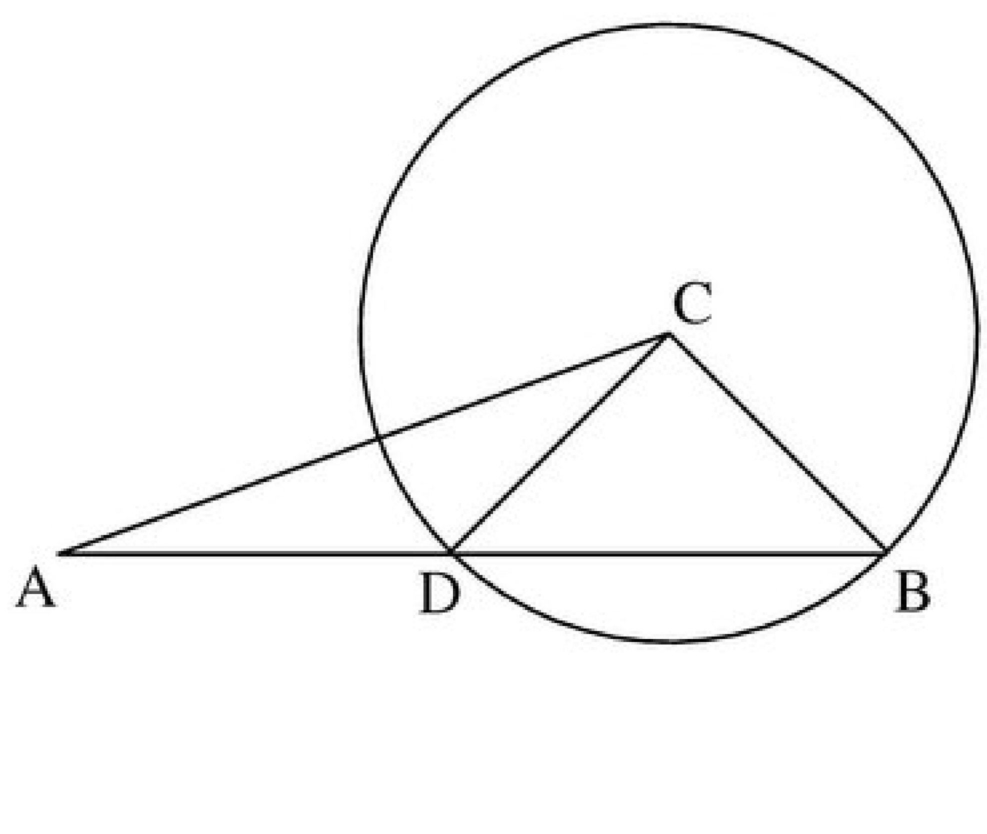
A. I and II only
B. I and III only
C. II and III only
D. I, II and III

11. It is given that $m \propto \sqrt{n}$. Find the percentage change in n such that m is increased by 25%.

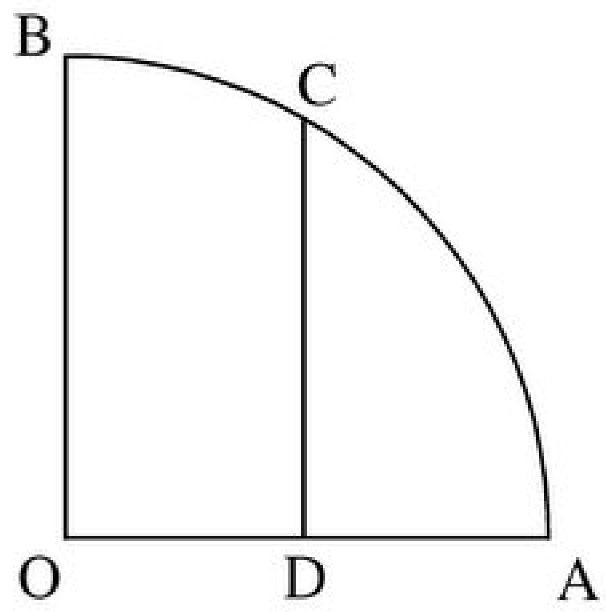
A. Increased by 5.25%
B. Increased by 25%
C. Increased by 50%
D. Increased by 56.25%

12. In the figure, C is the centre of the circle, if AB = 20, CB = 10 and D is the mid-point of AB, find the area of ΔADC.

A. 43.3 (square units)
B. 50.0 (square units)
C. 86.6 (square units)
D. 100 (square units)

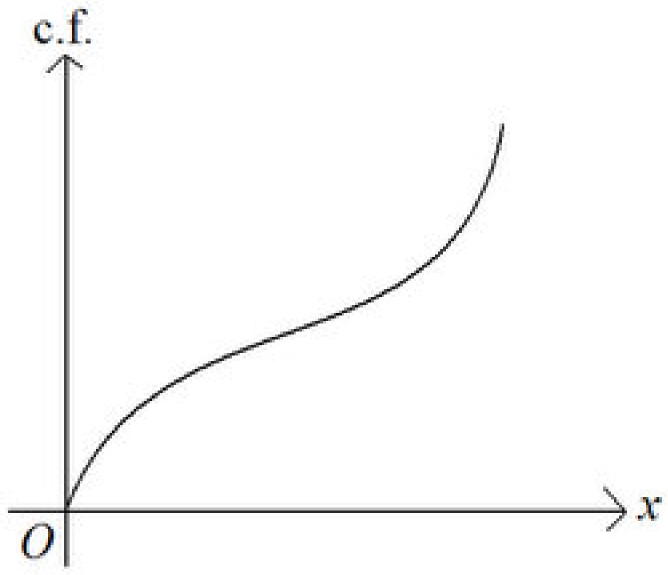


13. In the figure, the radius of sector AOB is $2cm$ and $\angle BOA = 90^{\circ}$. It is given that CD is the perpendicular bisector of AO, find the area of ADC.



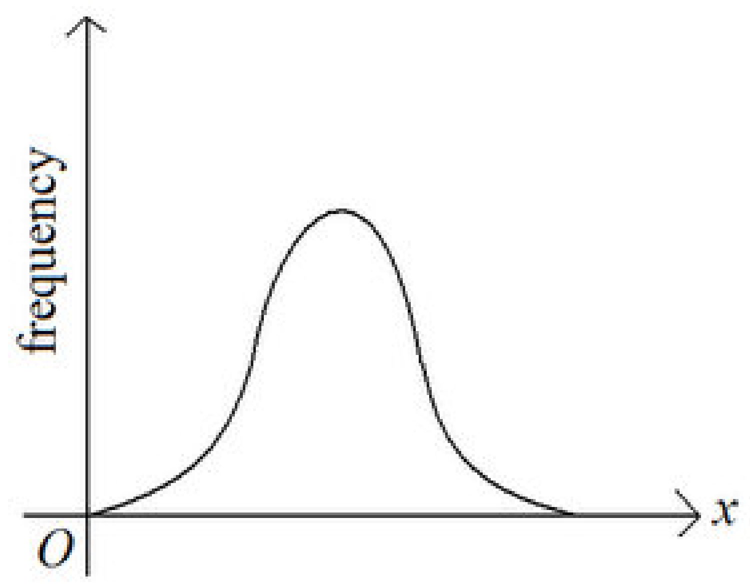
A. $0.866\,cm^2$ (correct to 3 decimal places)
B. $1.047\,cm^2$ (correct to 3 decimal places)
C. $1.142\,cm^2$ (correct to 3 decimal places)
D. $1.228\,cm^2$ (correct to 3 decimal places)

14. The figure shows the cumulative frequency curve of a certain distribution.

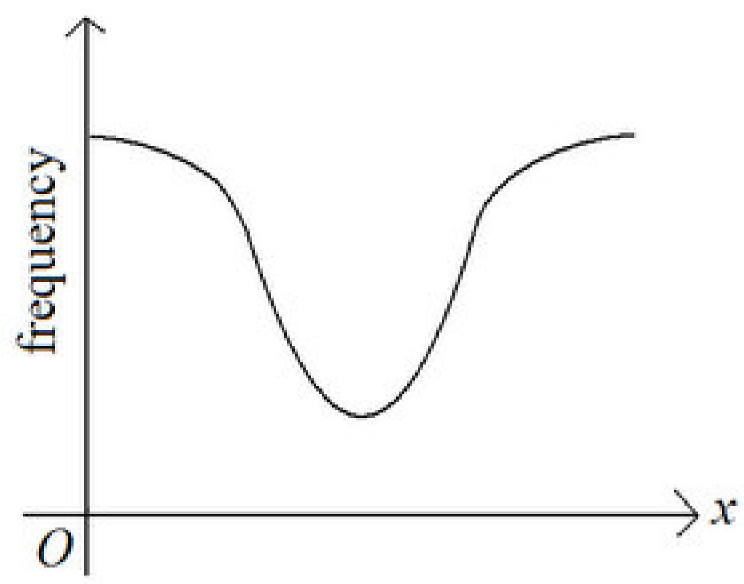


Which of the following frequency curves best represents the distribution?

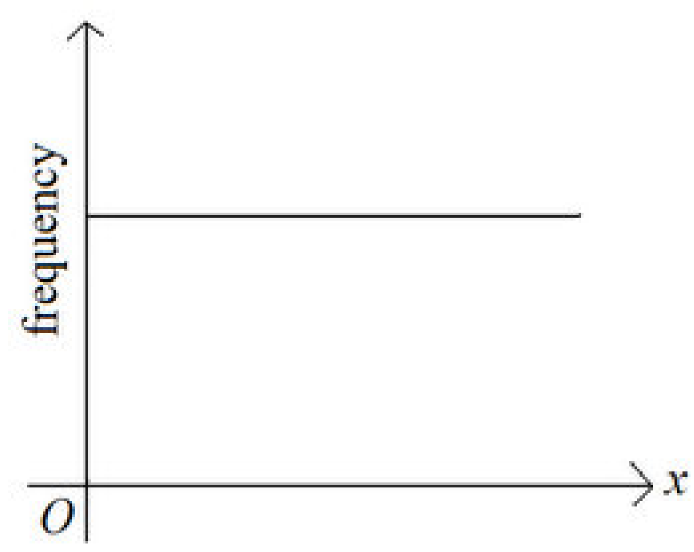
A.



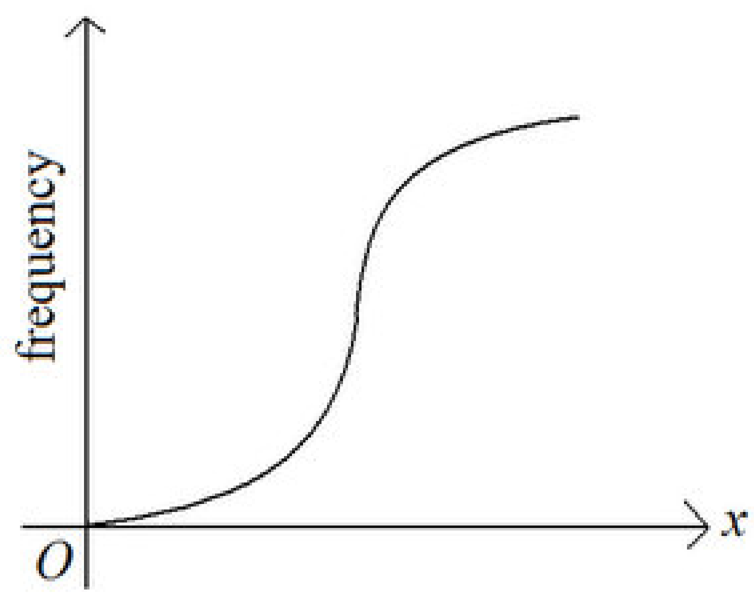
B.



C.



D.



15. If the point $A(a,b)$ is reflected about the line $x+y=0$ to the point A', what are the coordinates of A'?

A. $(-b,-a)$
B. $(-b,a)$
C. $(b,-a)$
D. (b,a)

16. If b and c are positive integers, then solve $x^2-bx+cx-bc\le 0$.

A. $-b\le x\le c$
B. $-c\le x\le b$
C. $x\ge b$ or $x\le -c$
D. $x\ge c$ or $x\le -b$

17. Let $\log 2=a$, $\log 3=b$ and $\log 7=c$, then $\log 315=$

A. $a+2b-c$.
B. $a-2b-c+1$.
C. $-a+2b+c+1$.
D. $-2a-2b+c-1$.

18. If a, b and c are consecutive positive integers, then

A. $a+b+c$ is even.
B. $a+b+c$ is odd.
C. abc is even.
D. abc is odd.

19. If α and β are the roots of the equation $2x^2+3x+4=0$, then $\dfrac{1}{\alpha+1}+\dfrac{1}{\beta+1}=$

A. 3.

B. − 3.

C. $\dfrac{1}{3}$.

D. $-\dfrac{1}{3}$.

20. If John walks 1 km/h faster, he will take $\dfrac{1}{6}$ hour less to travel 2 km, what is his original speed of walking?

A. 4 km/h

B. 3 km/h

C. 2 km/h

D. 1 km/h

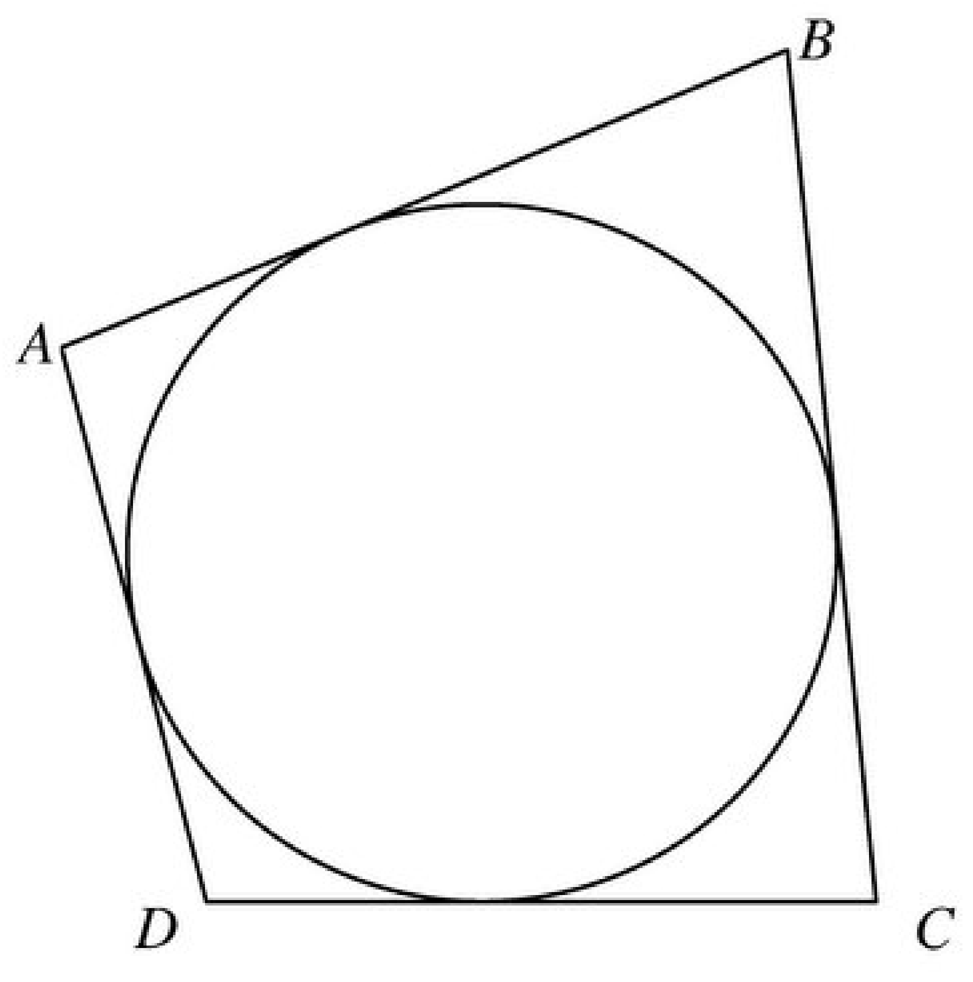
21. In the figure, a circle is inscribed in quadrilateral *ABCD*, where $BC=28$, $AD=22$. Find the perimeter of quadrilateral *ABCD*.

A. 88

B. 92

C. 96

D. 100



22. Find the number of distinct real roots of equation $(2x^2-5x)^2=(x^2-2x-3)^2$.

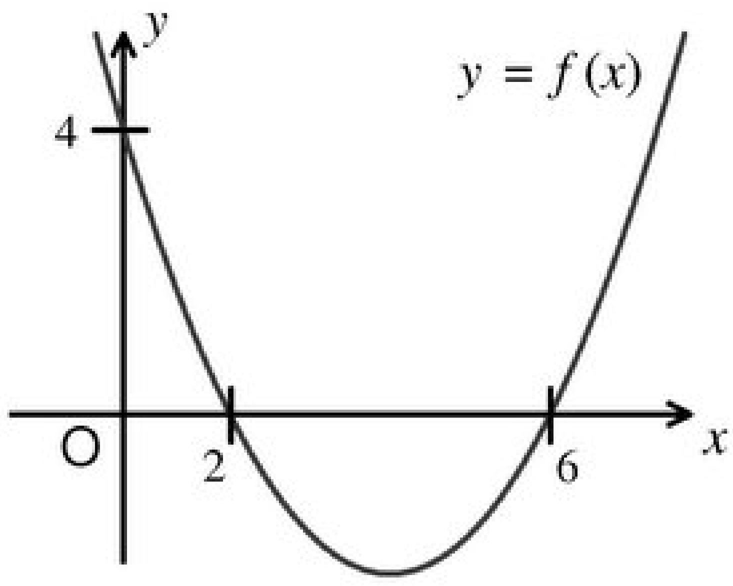
A. 0

B. 1

C. 2

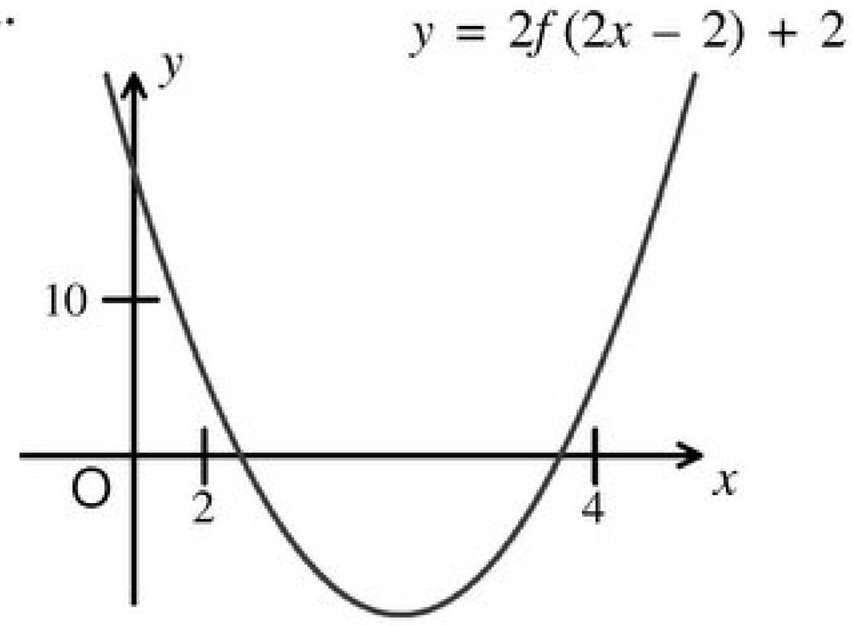
D. 4

23. The figure below shows the graph of $y = f(x)$.

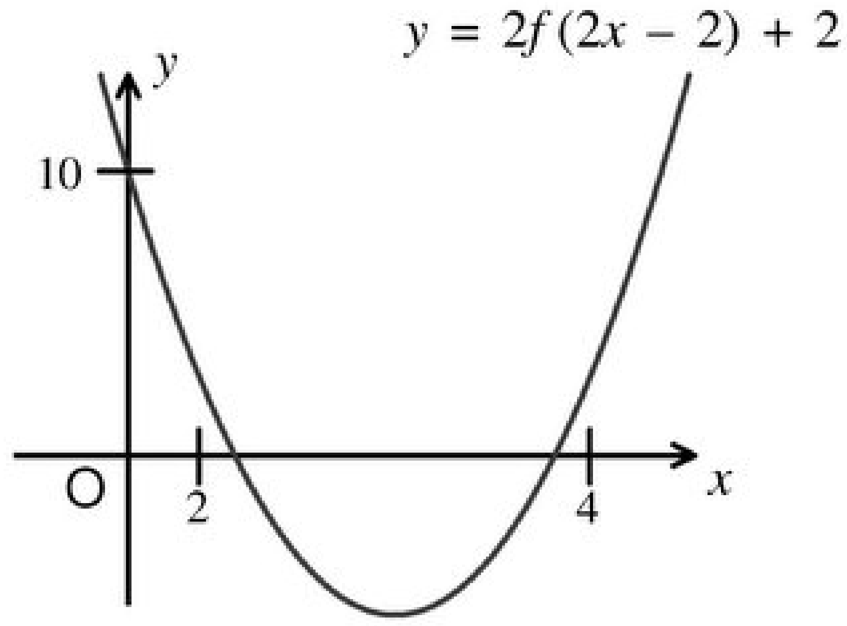


Which of the following may represent the graph of $y = 2f(2x-2)+2$?

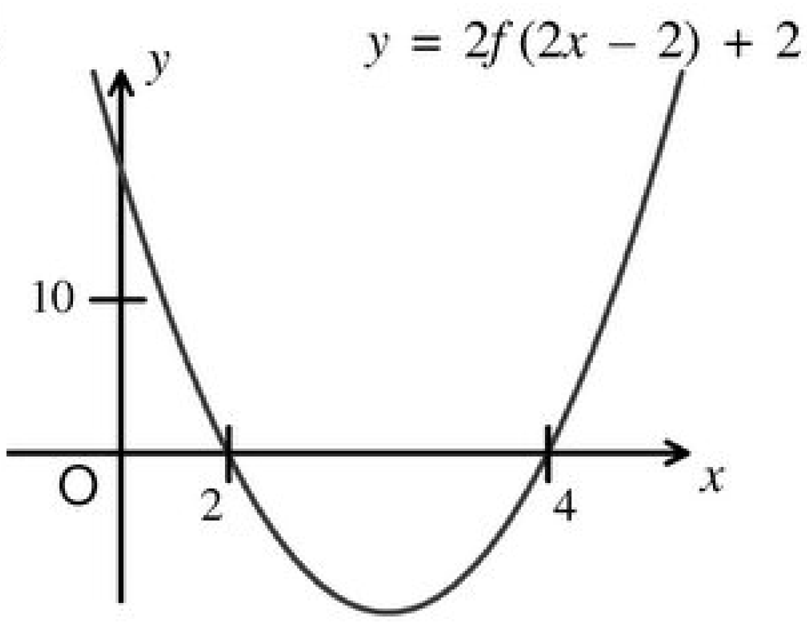
A.



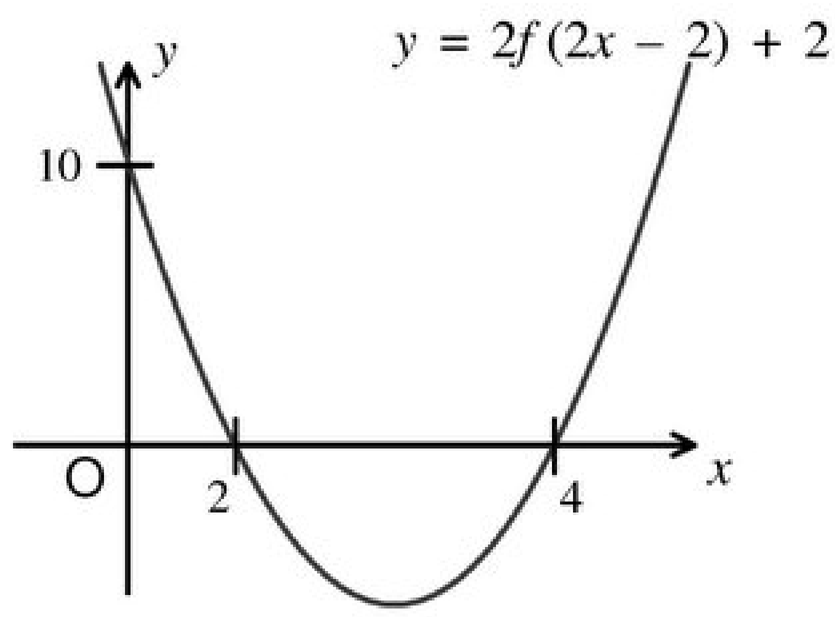
B.



C.



D.



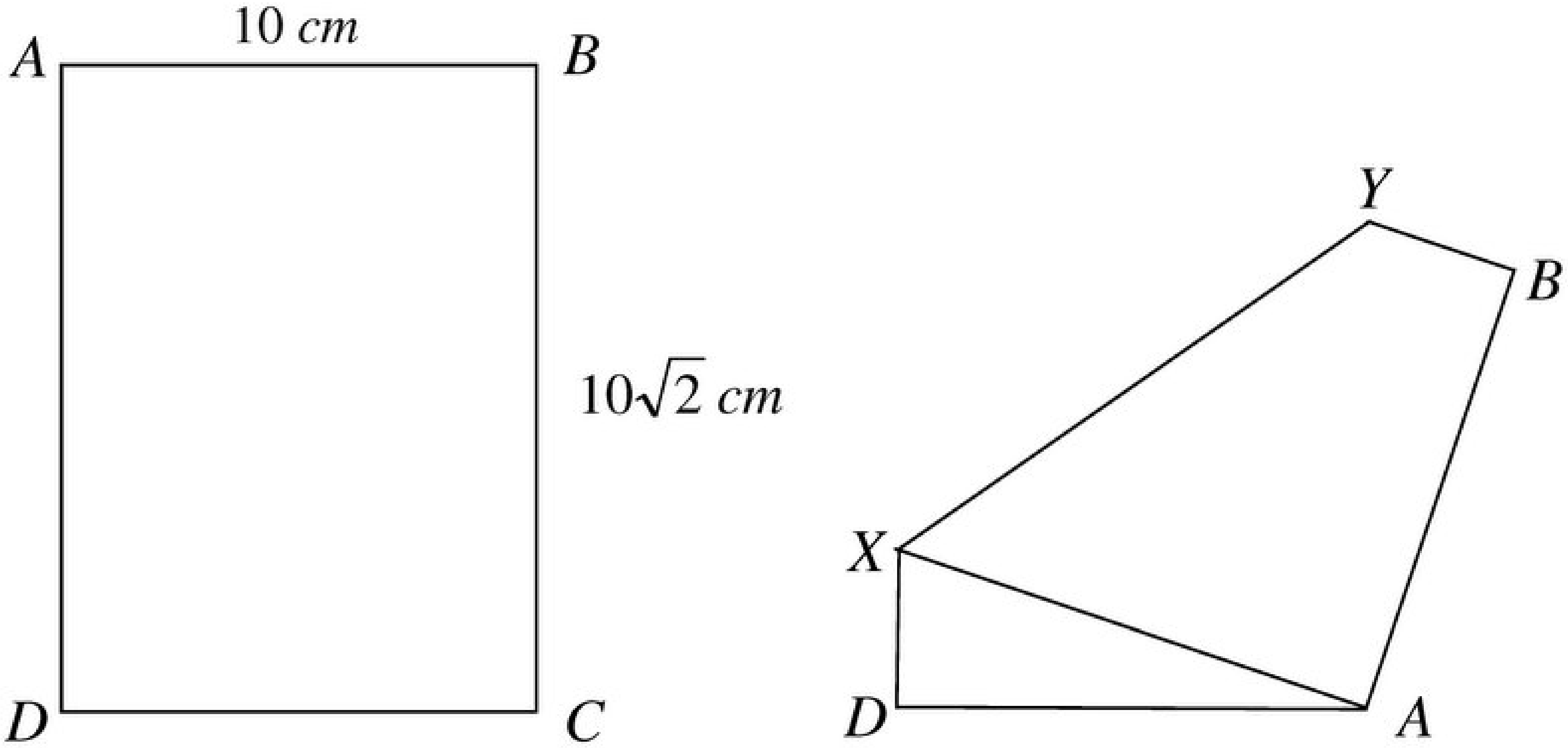
24. Which of the following has the greatest value?

A. $2012^{2011^{2012}}$
B. $2011^{2012^{2011}}$
C. $2012^{2011^{2011}}$
D. $2011^{2012^{2012}}$

25. If $x^3 - 3x^2 + 4$ is divisible by $x^2 + bx + c$, find the sum of the possible values of b.

A. -5
B. -2
C. -1
D. 0

26. In the figure, $ABCD$ is a sheet of paper in rectangular shape where $AB = 10$ cm and $BC = 10\sqrt{2}$ cm. The paper is folded along XY such that A is placed exactly on top of C. Find XD.

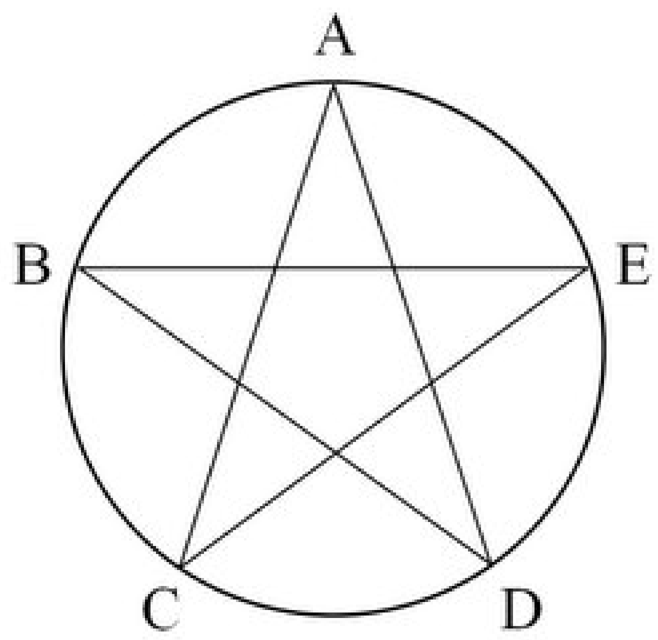


A. $\dfrac{10\sqrt{2}}{3}$ cm
B. $\dfrac{5\sqrt{2}}{2}$ cm
C. $2\sqrt{2}$ cm
D. Cannot be determined due to inadequate information

27. If $y^2 - x^2$ varies inversely as $\dfrac{1}{x^2} - \dfrac{1}{y^2}$. Which of the following is true ?

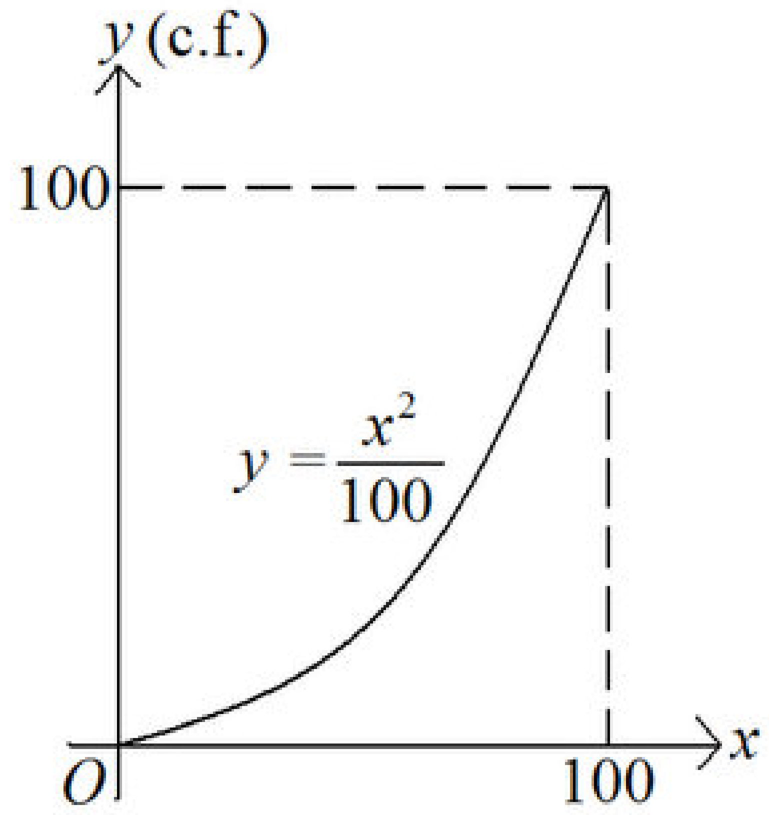
A. $xy \propto (x^2 + y^2)$
B. $xy \propto (x^2 - y^2)$
C. $x^2y^2 \propto (x^4 + y^4)$
D. $x^2y^2 \propto (x^4 - y^4)$

28. In the figure, a regular star ABCDE is inscribed in a circle. If AC = 10, find the diameter of the circle.



A. 10.515 (correct to 3 decimal places)
B. 11.180 (correct to 3 decimal places)
C. 12.361 (correct to 3 decimal places)
D. 16.180 (correct to 3 decimal places)

29. The figure below shows the cumulative frequency curve of a certain distribution.



Which of the following is/are correct?

I. Median = 50
II. Interquartile Range < 50
III. Mean > 50

A. I and II only
B. I and III only
C. II and III only
D. I, II and III

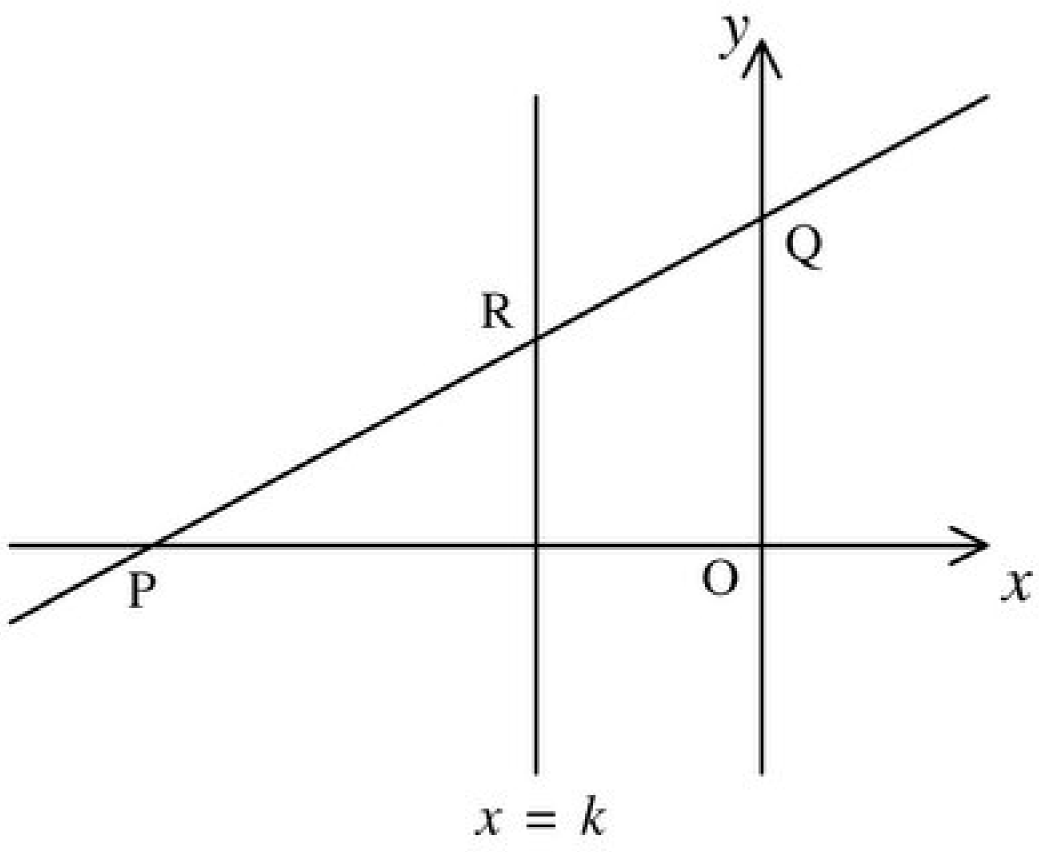
30. In the figure, P = (–5,0) and Q = (0,4). If the straight line PQ intersects the line $x = k$ at R, find the y-coordinate of R.

A. $\frac{20-4k}{5}$

B. $\frac{20+4k}{5}$

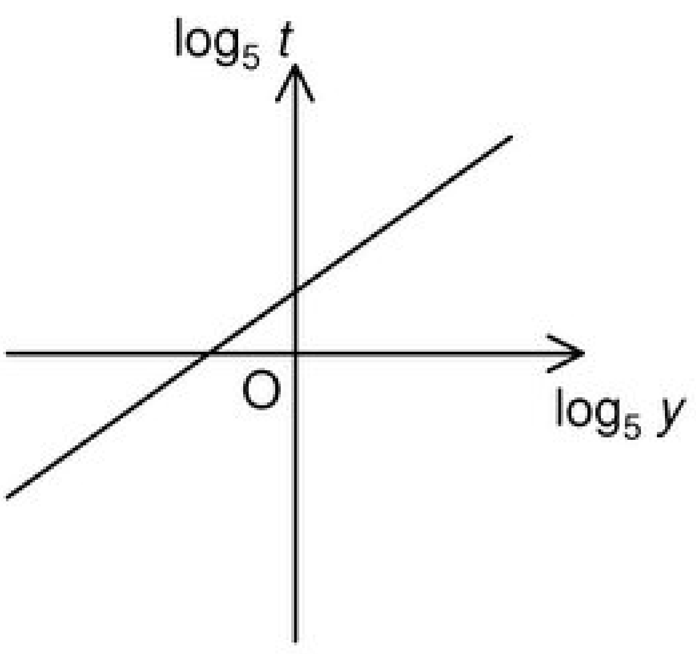
C. $\frac{25-5k}{4}$

D. $\frac{25+5k}{4}$

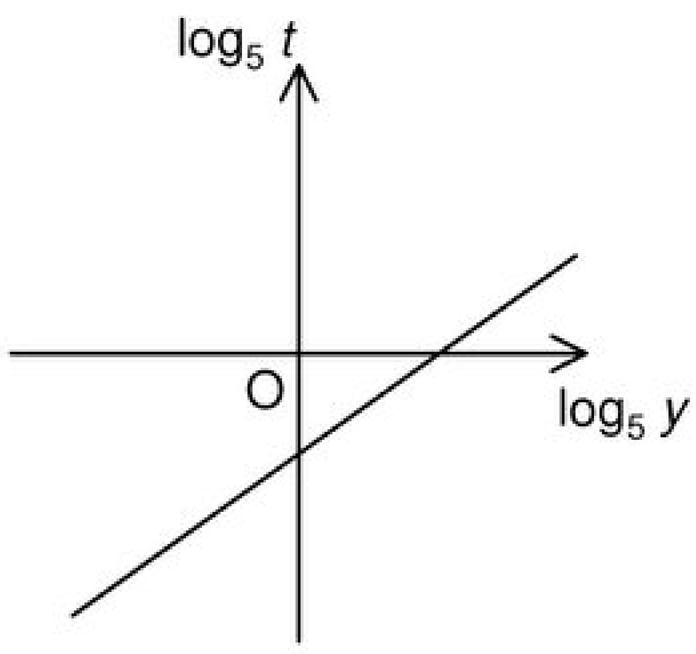


31. Given that $y = \frac{t^3}{25}$, which of the following graphs shows the linear relation between $\log_5 y$ and $\log_5 t$?

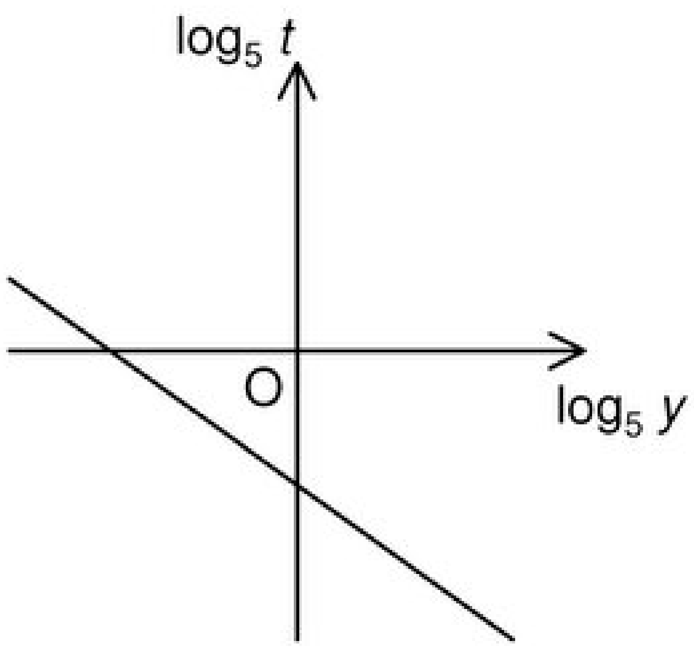
A.



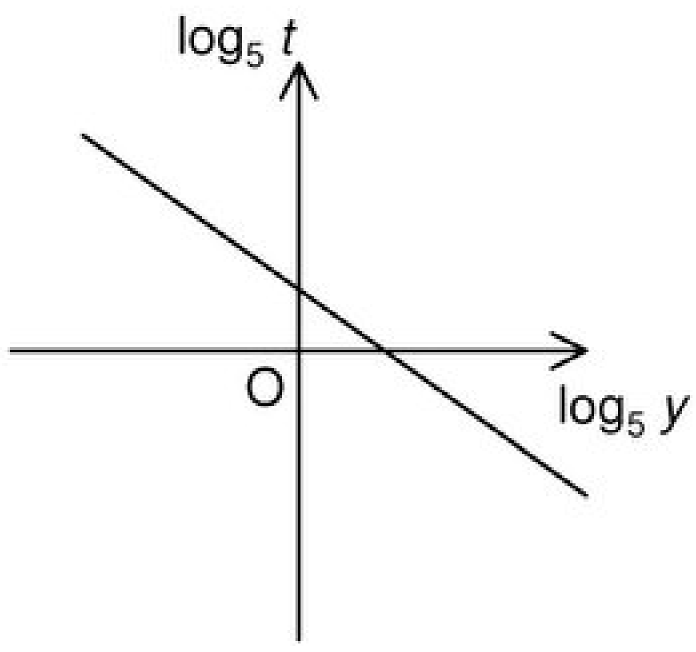
B.



C.



D.



32. Four different numbers are chosen from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. What is the probability that the chosen numbers have the smallest standard deviation?

A. $\frac{2}{105}$

B. $\frac{1}{210}$

C. $\frac{1}{35}$

D. $\frac{1}{30}$

33. Let $f(x) = x^2 + bx + c$, where b and c are real numbers. One of the roots of $f(x) = 0$ is $4 - 3i$, where $i = \sqrt{-1}$. Find the value of $b - c$.

A. –33
B. –34
C. –35
D. –36

34. The graph of $y = \sqrt{1-(x-2)^2} + 3$ is rotated about the origin counterclockwise 90°. Find the minimum y-value of points of the resulting graph.

A. 0
B. 1
C. 2
D. 3

35. The mean, median, and mode of real numbers 4, 4, 4, 6, 7, 18, x are not all equal. When they are arranged in ascending order, they form an arithmetic sequence. Find the sum of all possible values of x, correct to the nearest integer.

A. 6
B. 10
C. 14
D. 18

END OF PART A

PART B

ANSWER EITHER Section B1, B2 or B3 in this part.

Section B1

ANSWER FIVE QUESTIONS ONLY

36. [Arithmetic and geometric sequences and their summations]

An infinite table of integers is shown as follows:

$$\begin{array}{cccccc} 1 & 5 & 9 & 13 & 17 & \cdots \\ 4 & 9 & 14 & 19 & 24 & \cdots \\ 7 & 13 & 19 & 25 & 31 & \cdots \\ 10 & 17 & 24 & 31 & 38 & \cdots \\ 13 & 21 & 29 & 37 & 45 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

Each row in the table is an arithmetic sequence. Each column in the table is an arithmetic sequence.

(a) Find the first integer of the 2012^{th} row.

(b) Find the integer at the intersection of 2012^{th} row and 2012^{th} column.

37. [Arithmetic and geometric sequences and their summations]

In a geometric sequence of positive real numbers, the sum of the first 3 terms is 2 and the sum of the first 9 terms is 14.

(a) Find the possible value(s) of its common ratio.

(b) Find the sum of its first 12 terms.

38. [Permutation, combination and probability]

Three boys and seven girls are arranged to sit in a row of ten chairs.

(a) Suppose that the three boys sit together. In how many ways can the ten children choose their seats?

(b) Suppose that each boy sits between two girls. In how many ways can they choose their seats?

39. [Permutation, combination and probability]

Twelve identical badges are shared among 8 students.

(a) If each student gets at least one badge, find the number of possible ways of sharing the badges.

(b) If each student gets at least zero badge, find the number of possible ways of sharing the badges.

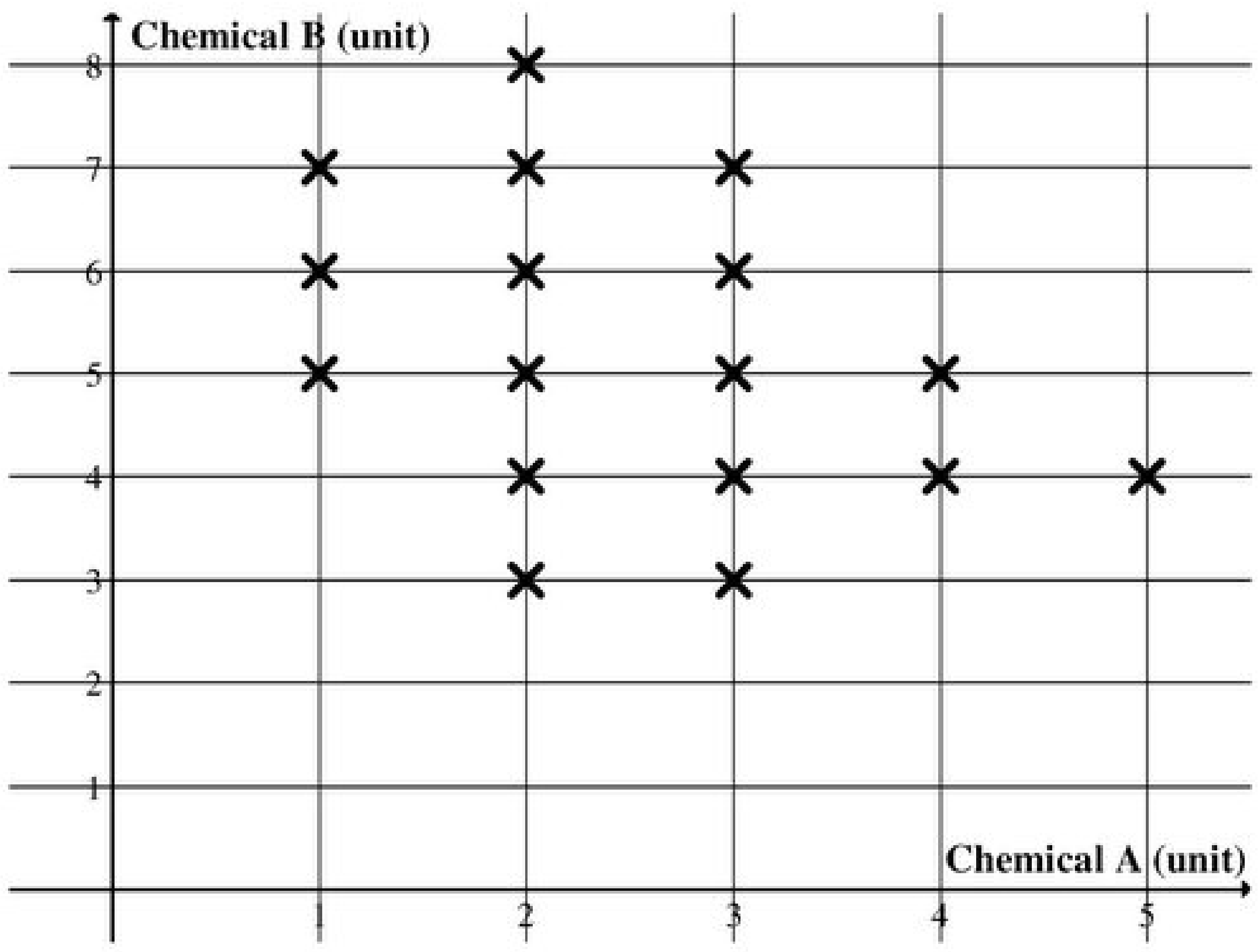
40. [Inequalities and linear programming]

Let $a = 300$, correct to the nearest hundred and $b = 10$, correct to the nearest ten.

(a) Find the range of values of $a + b$.

(b) Find the range of values of $\frac{a}{b}$.

41. [Inequalities and linear programming]



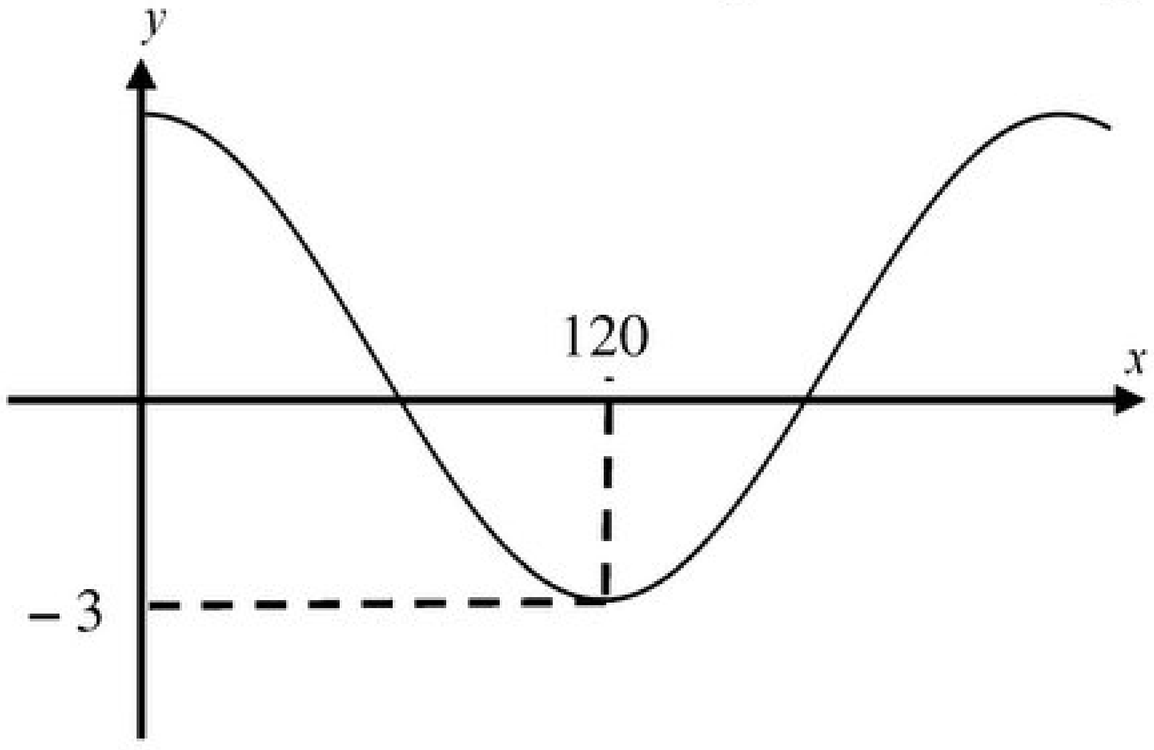
A scientist mixes chemical A and B to form a new product. Under certain constraints, the feasible combinations of two chemicals are shown by "x" in the figure. The cost of each unit of A and B are \$1 000 and \$3 000 respectively.

(a) Find the minimum cost of the product.

(b) The scientist finds that the data shown in above figure is wrong. The feasible combinations of two chemicals should include all integral quantities (in units) of A and B where the quantities of chemical A is within 1 to 5 (inclusive) and the quantities of chemical B is within 1 to 8 (inclusive).
If the scientist wants to keep the minimum cost of the product obtained in (a), how many total feasible combination(s) of two chemicals satisfies/satisfy the conditions?

42. [Trigonometry]

Let a be a constant and $-90° < \theta < 90°$. The figure shows the graph of $y = a\cos(x° + \theta)$.

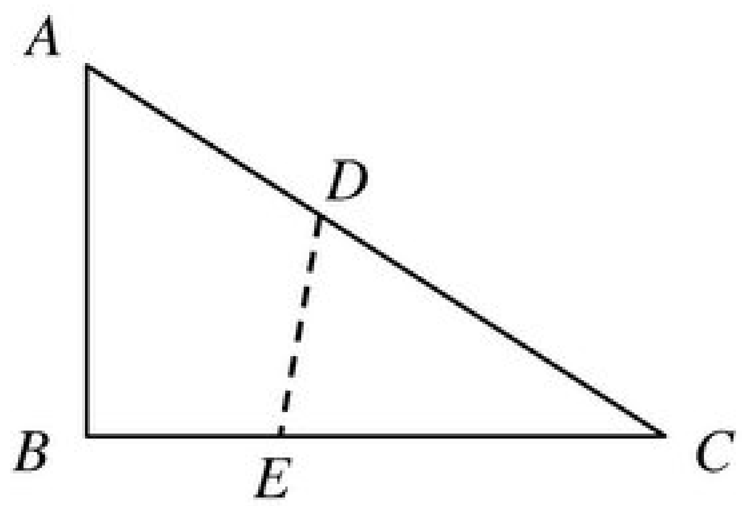


(a) Find the value of a.

(b) Find the value of y when $x = 960°$.

43. [Trigonometry]

In ΔABC, AB = 3 cm, BC = 4 cm, AC = 5 cm. Point D and E are drawn on AC and BC respectively, such that area of ΔDEC is equal to 6 cm^2. Let the length of DE, EC, CD be x cm, y cm, z cm respectively, and $\angle C = \theta$.



(a) Find an expression of yz in terms of θ.

(b) Find the minimum possible value of x.

44. [Equations of circle]

Find the equations of two circles with radius 5 units, passing through the point (1, 2) and touching the x-axis. Write your answers in (a) and (b) separately.

45. [Equations of circle]

Let the equations of circles C_1 and C_2 be $x^2 + y^2 + 10x - 2y + a = 0$ and $x^2 + y^2 - 14x + 28y + 81 = 0$ respectively, where a is a constant. The two circles touch each other externally at the point P.

(a) Find the value of a.

(b) Find the equation of the common tangent of C_1 and C_2 at the point P.

END OF PART B SECTION B1

Section B2

ANSWER FIVE QUESTIONS ONLY

36. [Binomial Expansion]

In the expansion of $\left(\frac{2}{x}+\frac{x^2}{4}\right)^9$, if the general term is $C_r^9(2)^k(x)^{-k}$,

(a) express k in terms of r and

(b) find the constant term.

37. [Binomial Expansion]

(a) Find the coefficient of x^r in the expansion of $(x+1)^{99}$, where r is an integer and $0 \le r \le 99$.

(b) Hence or otherwise, find the remainder when 71^{99} is divided by 1000.

38. [Exponential and logarithmic functions]

(a) Solve the exact value of x in equation $\ln(e^{2x}-9)-\ln 4048141=\ln(e^x+3)$.

(b) Using the result of (a), find the value of y if $\ln y=\frac{x}{2}$.

39. [Exponential and logarithmic functions]

The amount Q (in mg) of a piece of radioactive element is recorded after t days as follows:

t	0	1	2	3	4	5
$\ln Q$	7.80	7.57	7.34	7.11	6.88	6.65

(a) Suppose that Q and t can be modelled by $Q(t)=ka^t$, where $a>0$ and $a \ne 1$. Find a, correct to 3 significant figures.

(b) Find the value of t, correct to the nearest integer, when the amount of the radioactive element is $\frac{1}{4}$ of its initial value.

40. [Differentiation]

If $y=x^k+5x$, where k is a non-zero constant, find

(a) $\frac{d^2y}{dx^2}$ and (b) k if $2x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+15x=0$.

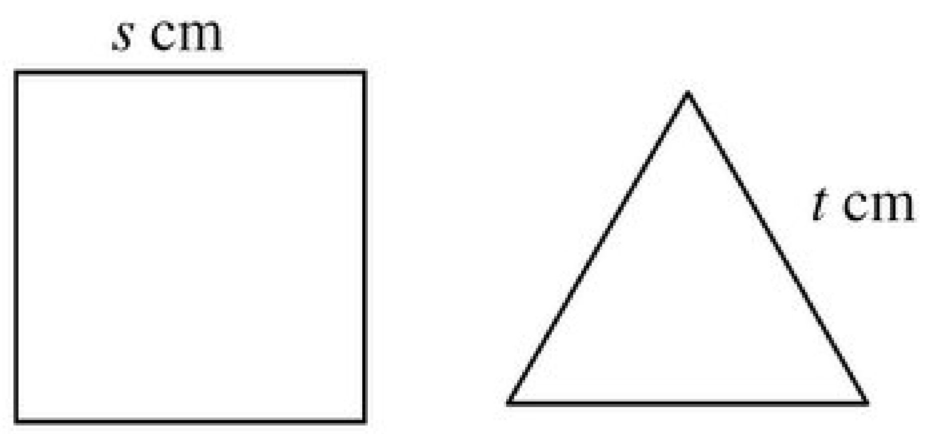
41. [Differentiation]

(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\sqrt{x}\ln x\right)$ when $x=1$.

(b) Hence, find $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}^{\sqrt{x}}$ when $x=1$.

42. [Differentiation]

The total area of a square and an equilateral triangle is 2012 cm^2. Let the length of a side of the square = s cm, the length of a side of the equilateral triangle = t cm, and $r = \frac{s}{t}$.

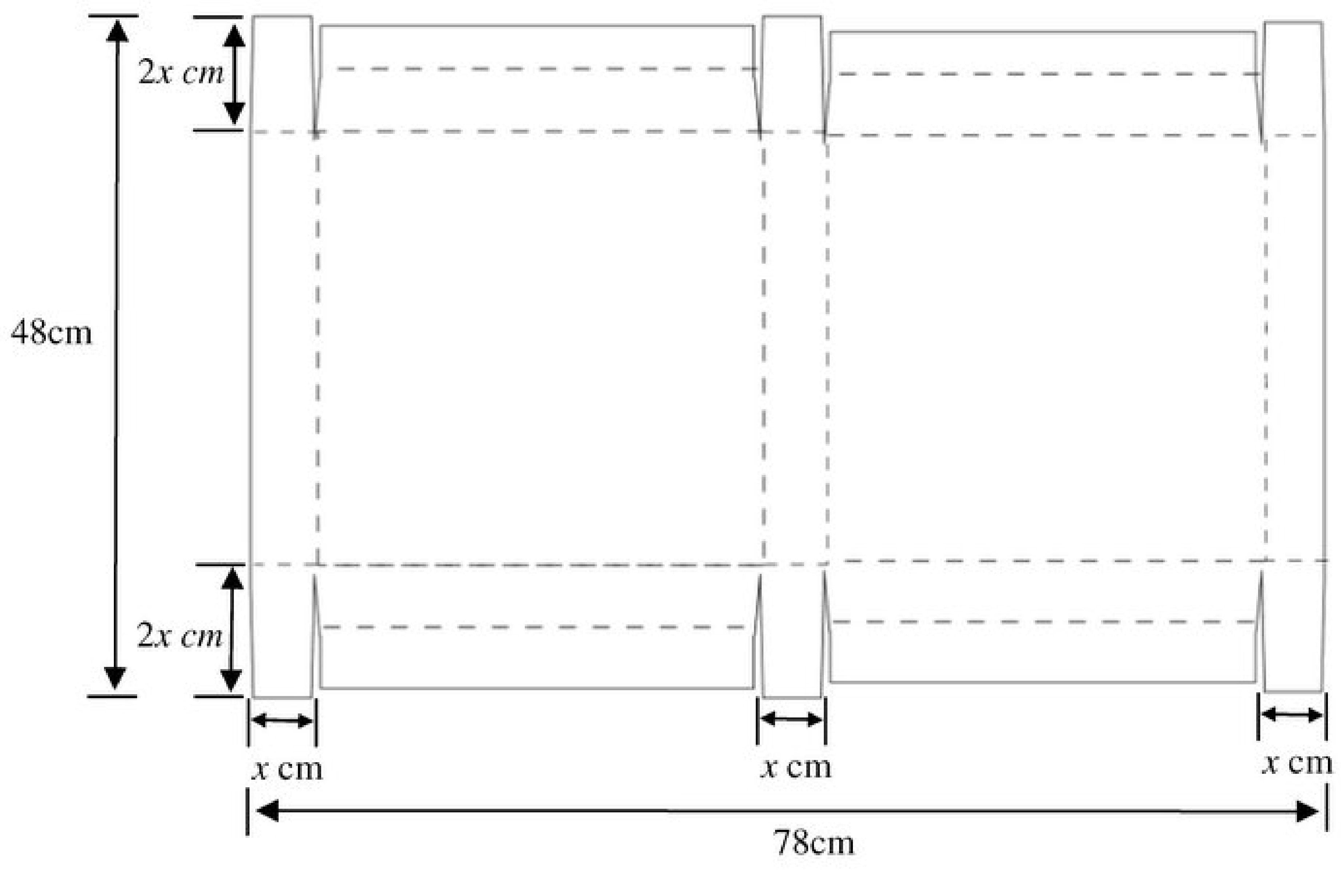


(a) If $s = r\sqrt{f(r)}$ and $t = \sqrt{f(r)}$, where $f(r)$ is a function of r,

find an expression for $f(r)$. (Give the answer in surd form if necessary.)

(b) Find the value of r such that the total perimeter of the square and triangle is maximized. (Give the answer in surd form if necessary.)

43. [Differentiation]



A rectangular piece of cardboard of length 78 cm and width 48 cm is cut to the shape shown in the figure. A pizza box of depth x cm with a lid is formed by folding along the dotted lines. Assume that all angles are right angles and the thickness of the cardboard is neglected.

Let V cm^3 be the volume of the pizza box.

(a) If $V = Ax^3 + Bx^2 + 1872x$, find A and B.

(b) If V is a maximum, find the value of x, correct to 3 significant figures.

44. [Integration]

Evaluate the following integrals:

(a) $\displaystyle\int_{2}^{4} (5x^2-1)\,\mathrm{d}x - \int_{2}^{-2} (5x^2-1)\,\mathrm{d}x$,

(b) $\displaystyle\int_{1}^{4} x\sqrt{x}\,\mathrm{d}x + \int_{4}^{25} t\sqrt{t}\,\mathrm{d}t$.

45. [Integration]

The rates of change of annual expenses of Mr Leung can be modelled by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{100e^{0.1t}}{t+20}$$

where E (in thousand dollars) is the total expenses of Mr. Leung and t is the number of years elapsed since 1 January 2008.

(a) Let $\tilde{E} = \int_{0}^{n} \dfrac{100e^{0.1t}}{t+20} dt$ be the total expenses of Mr. Leung from 1 January 2008 to 31 December 2011, what is the value of n?

(b) Use the trapezoidal rule with 4 subintervals to estimate the total expenses of Mr. Leung from 1 January 2008 to 31 December 2011. Correct the answer to the nearest integer.

END OF PART B SECTION B2

Section B3

ANSWER FIVE QUESTIONS ONLY

36. [Binomial Theorem]

In the expansion of $\left(\frac{2}{x}+\frac{x^2}{4}\right)^9$, if the general term is $C_r^9(2)^k(x)^{-k}$,

(a) express k in terms of r and

(b) find the constant term.

37. [Binomial Theorem]

(a) Find the coefficient of x^r in the expansion of $(x+1)^{99}$, where r is an integer and $0 \le r \le 99$.

(b) Hence or otherwise, find the remainder when 71^{99} is divided by 1000.

38. [Trigonometric functions]

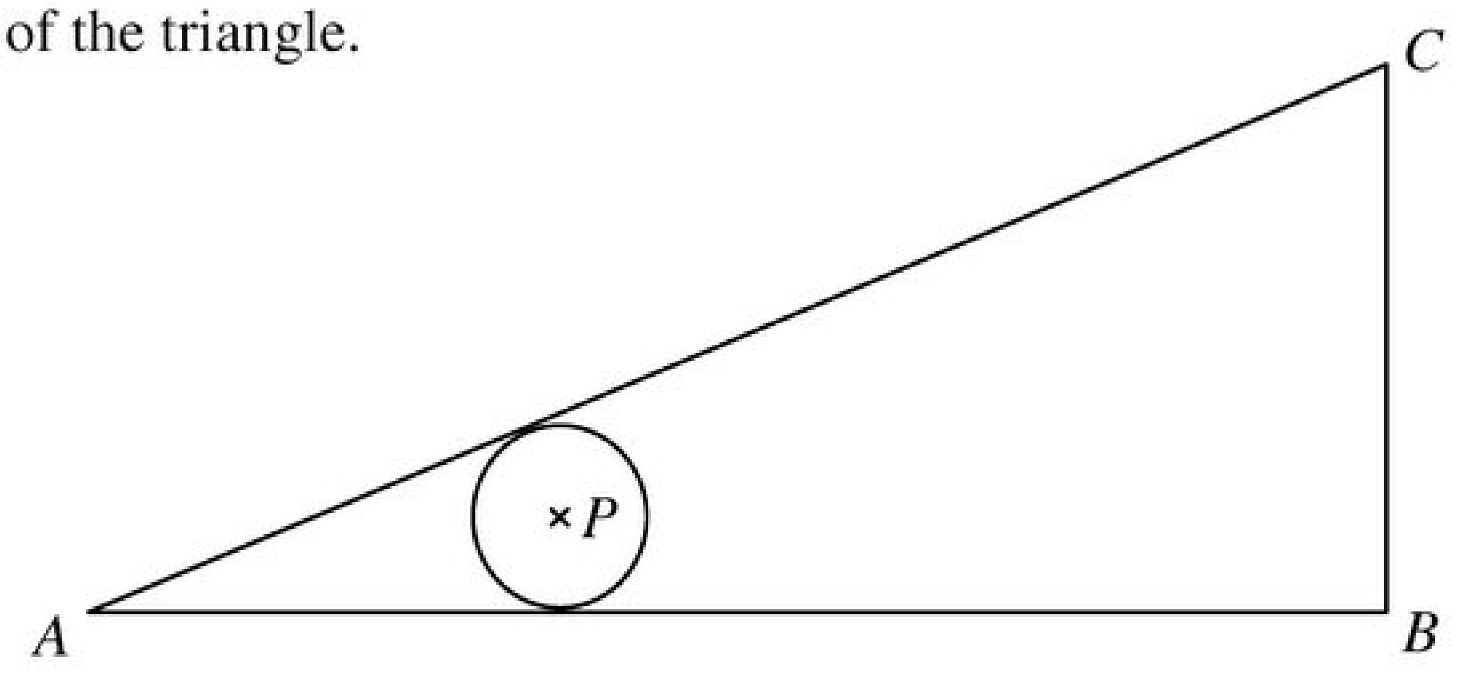
If $0 < B < A < \frac{\pi}{2}$ such that $\cos A \cos B = \frac{1}{4}$ and $\sin A \sin B = \frac{\sqrt{5}}{4}$.

(a) Find the value of $\cos(A-B)+\cos(A+B)$.

(b) Find the value of A in terms of π.

39. [Trigonometric functions]

In the figure, ΔABC is a right-angled triangle, $\angle B = 90°$, AB = 8 cm, BC = 6 cm.
A circular disk of radius 1 cm and centre P rolls inside ΔABC and is always tangent to at least one side of the triangle.



(a) What is the shape of locus of P?

(b) Let the area enclosed by the locus of $P = x$ cm^2. Find the exact value of x.

40. [Differentiation]

Given that $\tan y = x$.

(a) Express $\frac{dy}{dx}$ in terms of x.

(b) Find the value of $\frac{d^2y}{dx^2}$ when $y = \frac{\pi}{4}$.

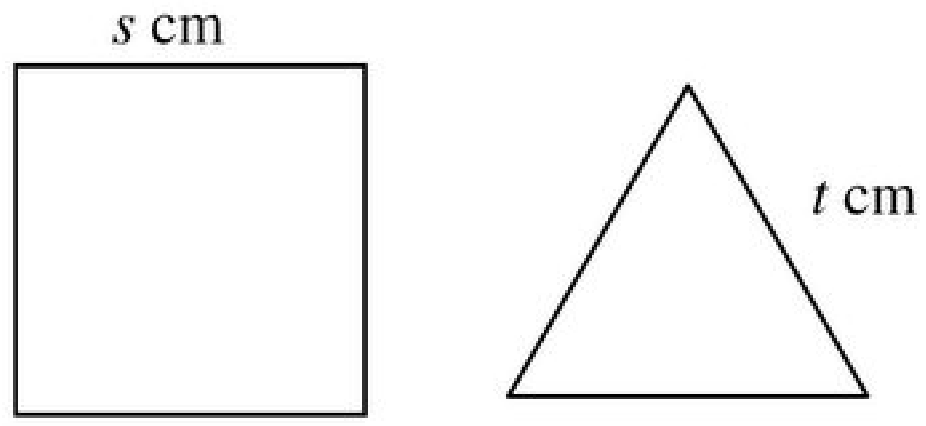
41. [Differentiation]

(a) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{2}\sqrt{x}\ln x\right)$ when $x=1$.

(b) Hence, find $\dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}^{\sqrt{x}}$ when $x=1$.

42. [Differentiation]

The total area of a square and an equilateral triangle is 2012 cm^2. Let the length of a side of the square = s cm, the length of a side of the equilateral triangle = t cm, and $r=\dfrac{s}{t}$.

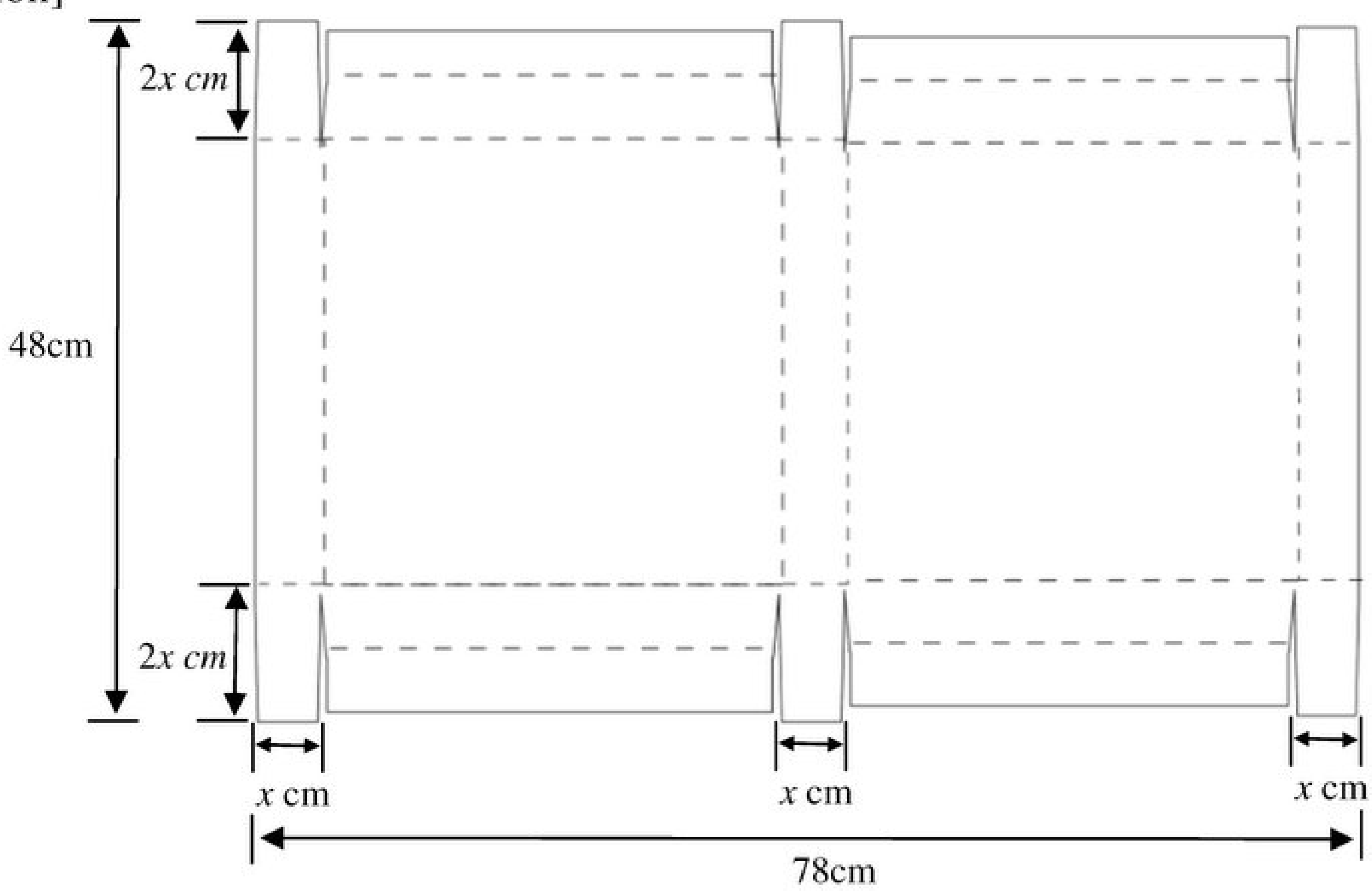


(a) If $s=r\sqrt{f(r)}$ and $t=\sqrt{f(r)}$, where $f(r)$ is a function of r,

find an expression for $f(r)$. (Give the answer in surd form if necessary.)

(b) Find the value of r such that the total perimeter of the square and triangle is maximized. (Give the answer in surd form if necessary.)

43. [Differentiation]



A rectangular piece of cardboard of length 78 cm and width 48 cm is cut to the shape shown in the figure. A pizza box of depth x cm with a lid is formed by folding along the dotted lines. Assume that all angles are right angles and the thickness of the cardboard is neglected.

Let V cm^3 be the volume of the pizza box.

(a) If $V=Ax^3+Bx^2+1872x$, find A and B.

(b) If V is a maximum, find the value of x, correct to 3 significant figures.

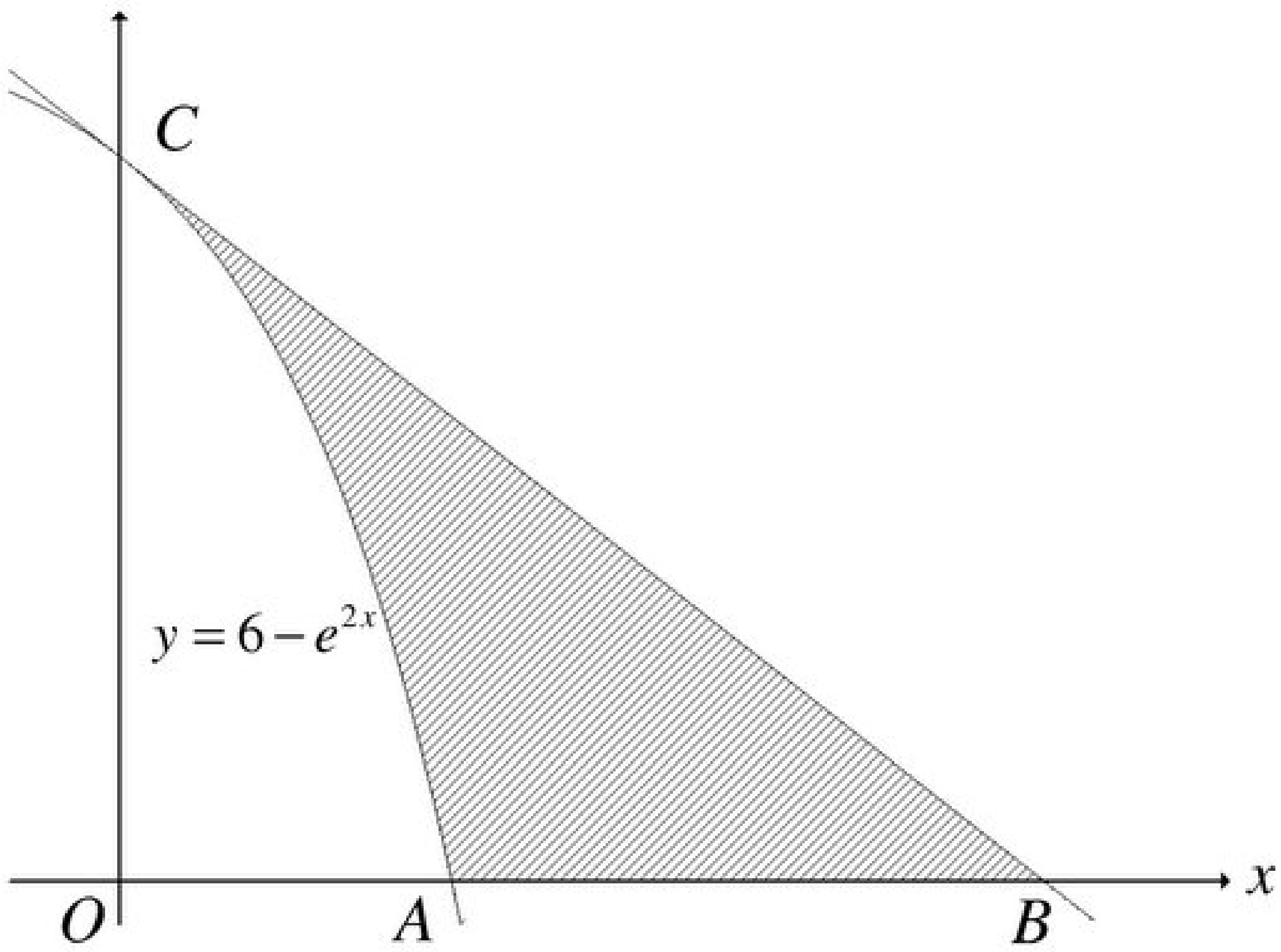
44. [Integration]

Evaluate the following integrals:

(a) $\int_2^4 (5x^2 - 1)\,\mathrm{d}x - \int_2^{-2} (5x^2 - 1)\,\mathrm{d}x$,

(b) $\int_1^4 x\sqrt{x}\,\mathrm{d}x + \int_4^{25} t\sqrt{t}\,\mathrm{d}t$.

45. [Integration]



The figure shows the graph of the curve $y = 6 - e^{2x}$ which cuts the x-axis and the y-axis at A and C respectively. The tangent to this curve at C cuts the x-axis at B.

(a) Find the coordinates of B.

(b) Find the area of the shaded region bounded by the curve, the tangent and the x-axis. Correct the answer to 3 significant figures.

END OF PART B SECTION B3

END OF PAPER