THE HONG KONG
POLYTECHNIC UNIVERSITY
香港理工大學

Secondary School Mathematics & Science Competition 2014

Mathematics

Date	: 17^{th} May 2014	Total no. of pages	: 22
Time allowed	: 9:30 am - 10:45 am (1hour 15 minutes)	Total marks	: 76
		(each MC question carries 2 marks)	

1. Write your Candidate Number, Centre Number, Name (both in English and Chinese), Name of School, Form, Date, Gender, Language and Subject in the spaces provided on the MC Answer Sheet and the Part B Answer Sheet.

2. When told to open this question paper, you should check that all the questions are there. Look for the words '**END OF PAPER**' after the last question.

3. Answer **ALL** questions in Part A. You are advised to use an **HB** pencil to mark your answers on the MC Answer Sheet.

4. You should mark only **ONE** answer for each question in Part A. If you mark more than one answer, you will receive **NO MARK** for that question.

5. Part B consists of Sections B(1), B(2) and B(3). Answer **ANY ONE** section. Answer **ANY FOUR** questions from your chosen section.

6. For Part B, answers may be exact values or mathematical expressions.

7. No mark will be deducted for wrong answers.

8. The diagrams in the paper are not necessarily drawn to scale.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

PART A

Answer all questions. Choose the best answer for each question.

1. If $a+\frac{1}{a}=3$, then $a^{3}+a^{-3}$ =

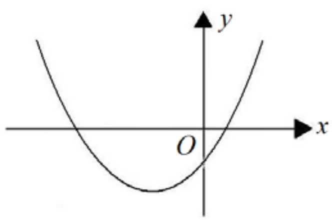
   A. 12.
   B. 18.
   C. 27.
   D. 36.

2. Let k be a real constant. How many distinct real roots does the equation $x^2+k(1-x)-2=0$ have?

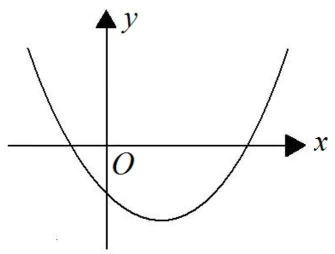
   A. 2
   B. 1
   C. 0
   D. Cannot be determined as the value of k is not known

3. Let a, b, and c be real numbers, and $abc>0$. Which of the following is a possible graph of the function $f(x)=ax^2+bx+c$?

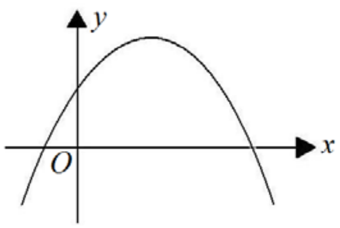
   A.



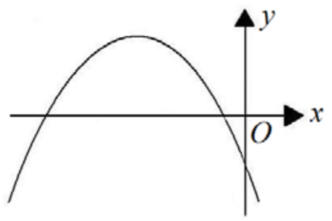
   B.



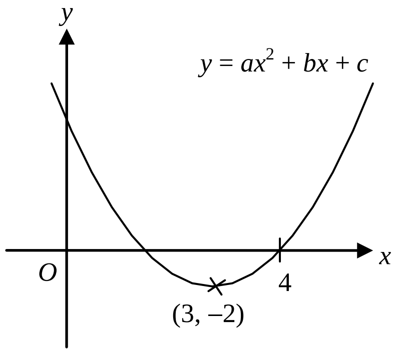
   C.



   D.



4. The figure shows the graph of $y = ax^2 + bx + c$, where a, b, c are real numbers. The coordinates of the vertex of the graph is (3, –2). The value of c is



A. –2.
B. 7.
C. 11.
D. 16.

5. Given that $f(x-2) = x^2 - 2x + 3$. Find $f(2)$.

A. –1
B. 3
C. 7
D. 11

6. It is given that $2^h = 5^k = 100^p$, where h, k and p are non-zero real numbers. Find the value of $\frac{p}{h} + \frac{p}{k}$.

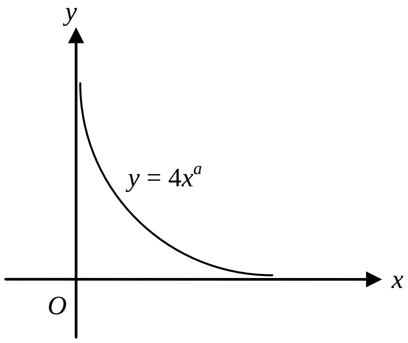
A. $\frac{1}{2}$

B. 1

C. $\log 7$

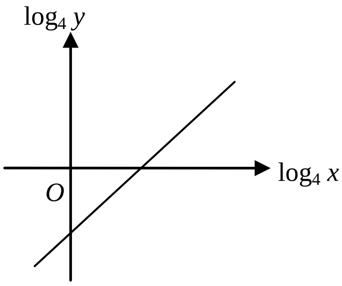
D. $\frac{1}{2}\log 7$

7. The figure shows the graph of $y = 4x^a$, where a is a constant.

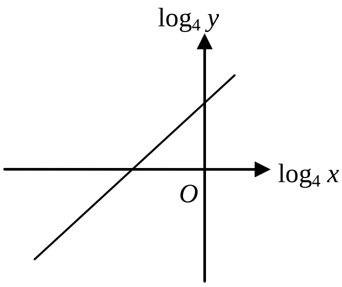


Which of the following graphs represents the relationship between $\log_4 x$ and $\log_4 y$?

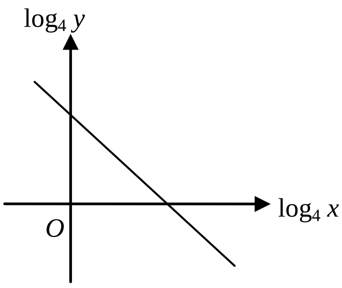
A.



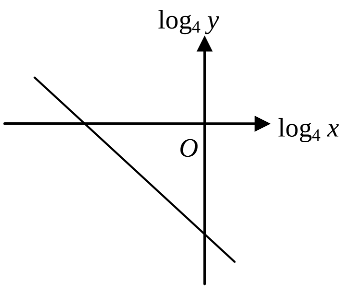
B.



C.



D.



8. Let the roots of the equations $\log_{27} x-\left(\frac{1}{27}\right)^{x}=0$ and $\log_{\frac{1}{27}} x-\left(\frac{1}{27}\right)^{x}=0$ be α and β respectively. Which of the following is true?

A. $0<\alpha\beta<1$
B. $\alpha\beta=1$
C. $1<\alpha\beta<3$
D. $\alpha\beta\geq 3$

9. Let $f(x)=x^3+ax^2+bx+c$, where a , b and c are real numbers. If $f(-2)=f(1)=f(3)=0$, find the value of c .

A. -6
B. -2
C. 2
D. 6

10. Let p, q and r be real numbers. If $x^4+4x^3+6px^2+4qx+r$ is divisible by x^3+3x^2+9x+3, then $(p+q)r=$

A. – 27.
B. – 18.
C. 15.
D. 45.

11. When a polynomial $f(x)$ is divided by $(x-1)$, the remainder is 3. When $f(x)$ is divided by $(x-3)$, the remainder is 5. Find the remainder when $f(x)$ is divided by $(x-1)(x-3)$.

A. $x+2$
B. $x-2$
C. $2x+1$
D. $2x-1$

12. Solve the equation $\left(\dfrac{2}{x}+5\right)^2=\left(\dfrac{2}{x}+5\right)$.

A. $-\dfrac{2}{5}$

B. $-\dfrac{5}{2}$

C. $-\dfrac{2}{5}$ or $-\dfrac{1}{2}$

D. $-\dfrac{5}{2}$ or -2

13. If $0^\circ \le x < 360^\circ$, the number of roots of the equation $2\cos^2 2x+\cos 2x=1$ is

A. 3.
B. 4.
C. 5.
D. 6.

14. Let x and y be two variables. The table below shows some corresponding values of x and y.

x	1	2	4	8
y	2	$\sqrt{2}$	1	$\sqrt{\dfrac{1}{2}}$

Find the positive value of y if $x=256$.

A. $\dfrac{1}{8}$

B. $\dfrac{1}{4}$

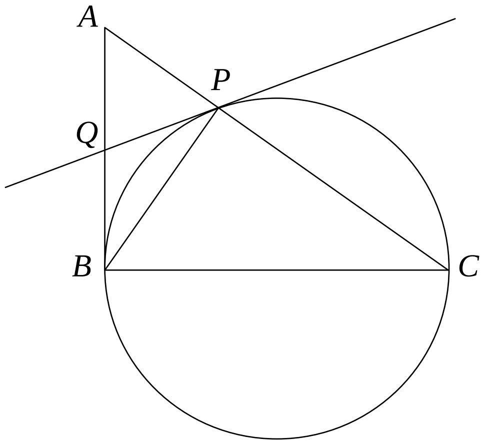
C. 4

D. 8

15. It is given that z varies partly as x^2, and partly as y. When $x=y=1$, $z=2$. When $x=y=2$, $z=9$. Find the value of z when $x=y=4$.

A. 16
B. 38
C. 76
D. 128

16. In the figure, BC is a diameter of the circle. Straight line AQB is perpendicular to BC. AC intersects the circle at P. PQ is tangent to the circle. Which of the following are true?



I. PQ bisects AB

II. $\triangle APQ$ is an isosceles triangle

III. $\angle BAC = \angle CBP$

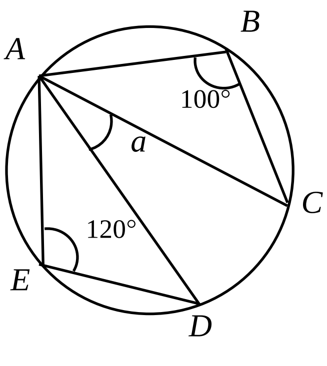
A. I and II

B. I and III

C. II and III

D. I, II and III

17. In the figure, $ABCDE$ is a circle. $\angle ABC = 100°$, $\angle AED = 120°$, and $\angle DAC = a$. Find the value of a.



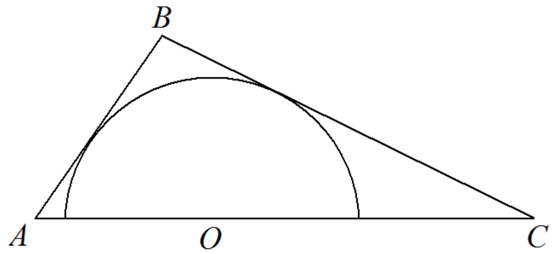
A. 40°

B. 50°

C. 60°

D. 70°

18. In the figure, a semicircle is inscribed inside $\triangle ABC$. O is the centre of the semicircle. $AB = 12$, $BC = 18$ and $AC = 25$. A, O, C are collinear. AB and BC are tangent to the semicircle. Find the length of AO.



A. 10
B. 11
C. 12
D. 13

19. Let m, c be real numbers, where $m \neq 0$. The line L: $y = mx + c$ is rotated 90° counterclockwise about its x-intercept to line L_1. The equation of L_1 is

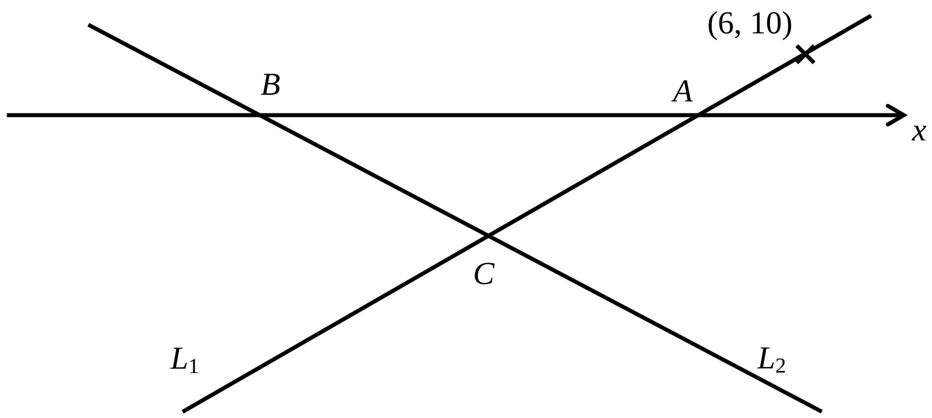
A. $y = \frac{1}{m}x + \frac{c}{m^2}$.

B. $y = -\frac{1}{m}x + \frac{c}{m^2}$.

C. $y = \frac{1}{m}x - \frac{c}{m^2}$.

D. $y = -\frac{1}{m}x - \frac{c}{m^2}$.

20. The straight line L_1 passes through (6, 10) and cuts the x-axis at A. Another straight line $L_2 : x + 2y + 20 = 0$ cut the x-axis at B. L_1 and L_2 intersect at C. If $AC = BC$, find the coordinates of C.



A. (–3.2 , –8.4)
B. (–10 , –5)
C. (–17 , –1.5)
D. (–18 , –1)

21. Let m, n be real numbers. $A(a,b)$ and $C(c,d)$ are two distinct points on the straight line $y=mx+n$. If $a>c$, then $AC=$

A. $(a-c)\sqrt{1+m^2}$.

B. $\dfrac{a-c}{\sqrt{1+m^2}}$.

C. $(a-c)(1+m^2)$.

D. $\dfrac{a-c}{1+m^2}$.

22. The variance of $x-4$, $x-2$, x, $x+2$ and $x+4$ is

A. 0.
B. 1.
C. 2.
D. 8.

23. Consider the real numbers 3, 4, 8, 11, a, b. It is given that the mean of the numbers is 6. If the standard deviation of the numbers is the smallest possible, what is the product of a and b?

A. 16
B. 21
C. 24
D. 25

24. Figure A shows a hemi-spherical container of inside radius 5 cm. It is filled with water. A solid cone of base radius 4 cm and slant height 10 cm is inserted into the container right side up as shown in Figure B, causing some water to overflow. Find the volume of water remaining in the container correct to 3 significant figures.

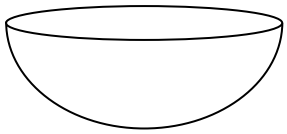


Figure A

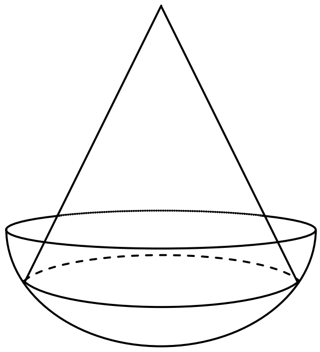
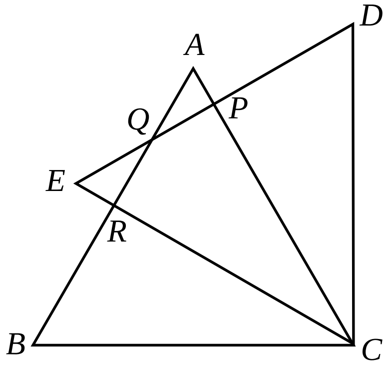


Figure B

A. 152 cm^3
B. 153 cm^3
C. 154 cm^3
D. 155 cm^3

25. In the figure, $\triangle ABC$ and $\triangle CDE$ are equilateral triangles. DE intersects AC and AB at P and Q respectively, and EC intersects AB at R. $DC \perp BC$, and $BC = 2$. Find the area of quadrilateral $PQRC$.



A. $3\sin 15°$

B. $3\tan 15°$

C. $\dfrac{3\sin 30°}{2}$

D. $\dfrac{3\tan 30°}{2}$

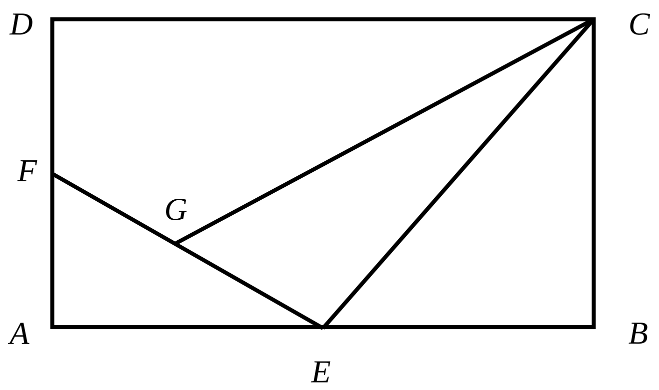
26. If A is a positive integer such that $\frac{1}{1\times 3}+\frac{1}{3\times 5}+...+\frac{1}{(A+1)(A+3)}=\frac{12}{25}$, find the value of A.

A. 19
B. 20
C. 21
D. 22

27. If p, q are prime numbers and $p+5q=97$, evaluate q^2-p.

A. 107
B. 239
C. 317
D. 359

28. In the figure, $ABCD$ is a rectangle. AB = 10, BC = 6. E, F are mid-points of AB and AD respectively. G is a point on EF, and FG = 3GE. Find the area of $\triangle CEG$.



A. $\frac{45}{8}$

B. $\frac{49}{8}$

C. $\frac{55}{8}$

D. $\frac{57}{8}$

29. There are 15 questions in a test. 5 marks are given for each correct answer. 1 mark is deducted for each wrong answer. No mark is deducted for not answering the question. There are m possible total marks. Find m.

A. 70
B. 79
C. 81
D. 99

30. In base 26, let the letters A to Z represent the digits 0 to 25 respectively. Let $\alpha = HAPPY_{26} \times SAD_{26}$, where $HAPPY_{26} = 7 \times 26^4 + 15 \times 26^2 + 15 \times 26 + 24$ and $SAD_{26} = 18 \times 26^2 + 3$. How many digits does α have when α is expressed in base 2?

A. 34
B. 35
C. 36
D. 37

END OF PART A

PART B

Answer ANY ONE SECTION from Sections B(1), B(2) or B(3).

SECTION B(1)

Answer any FOUR questions.

31. [Arithmetic and Geometric Sequences and their Summations]

Let a_n be the n-th term of an arithmetic sequence. It is given that $a_3 = -1$ and $a_6 + a_9 = -11$.

(a) Express a_n in terms of n.

(b) Express $a_1 + \frac{a_2}{2} + \frac{a_3}{4} + \cdots + \frac{a_n}{2^{n-1}}$ in terms of n.

32. [Arithmetic and Geometric Sequences and their Summations]

In Figure B1-1, an object travels along a spiral locus. In each pass, the object makes one horizontal move and then one vertical move. The initial distances of the moves in the first pass are both 1. Each move of each subsequent pass is twice as far as those in the previous pass.

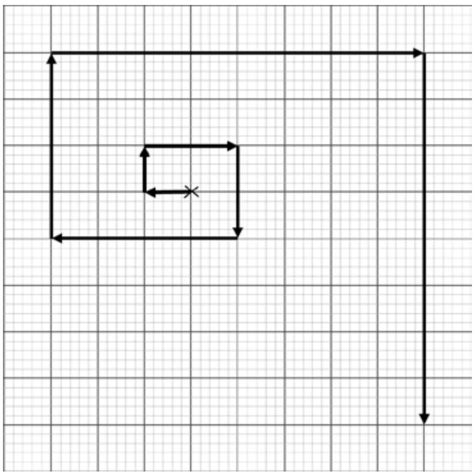


Figure B1-1

(a) Find the distance of the horizontal move in the 5th pass.

(b) Find the distance of the vertical move in the 6th pass.

(c) Find the total distance travelled by the object when it completes the 8th pass.

33. [Equations of Circles]

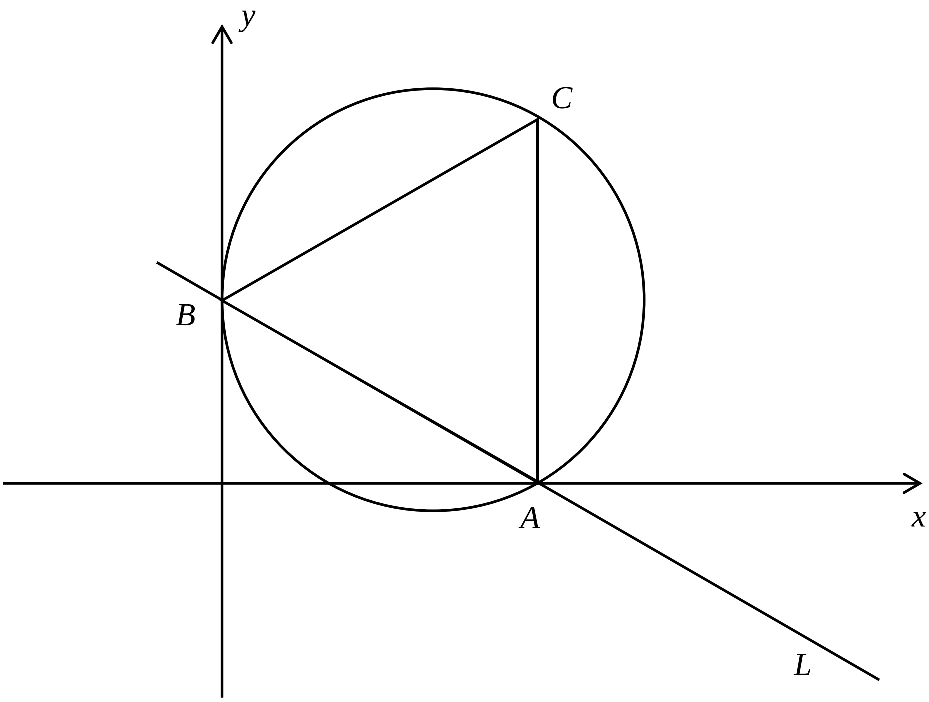


Figure B1-2

In Figure B1-2, straight line $L: \sqrt{3}x + 3y - 3 = 0$ cuts the x-axis and y-axis at A and B respectively. A circle passes through A, B and C, and $\triangle ABC$ is an equilateral triangle.

(a) (i) Find the coordinates of C.

(ii) Find the radius and the coordinates of the centre of the circle.

(b) Find the equation of the circle in centre-radius form.

34. [Locus]

In Figure B1-3, C_1 is a circle with centre $A(8, 4)$, radius 2, and C_2 is a circle with centre $B(4, 6)$, radius 1. PQ is a common tangent to C_1 and C_2. AB and PQ are produced to meet at C. It is assumed that the x-coordinate of P is larger than 8.

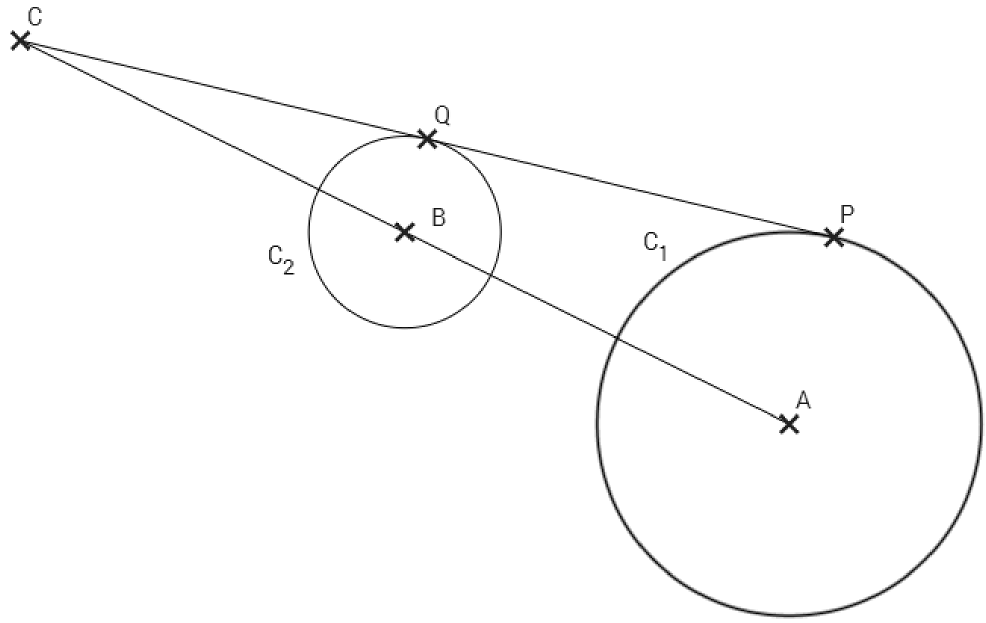


Figure B1-3

(a) Find the coordinates of C.

(b) Find the slope of PQ. Give the answer correct to 3 decimal places.

(c) Use the result of (b) to find $\angle PCA$. Give the answer correct to 1 decimal place.

35. [Permutation, Combination and Probability]

9 seats are arranged in a row for 8 students from 3 different classes. There are 2 students from 5A, 3 from 5B and 3 from 5C.

(a) (i) Find the number of ways in which these 8 students can take the seats.

(ii) Find the number of ways in which these 8 students can take the seats if the 2 students from 5A must sit next to each other. (Two students sitting next to each other means there's no occupied/unoccupied seat between them.)

(b) The 9 seats are now rearranged into 3 separate rows of 3 seats each. Find the number of ways in which these 8 students can take the seats, if the 2 students from 5A must sit next to each other.

36. [Permutation, Combination and Probability]

7 boys and 5 girls are randomly divided into four groups of equal size. Find the probability that

(a) one of the groups consists of all girls;

(b) there is at least one girl in each group.

END OF SECTION B(1)

SECTION B(2)

Answer any FOUR questions.

37. [Binomial Expansion, Exponential and Logarithmic Functions]

It is given that $(4+3x)^4(1-x)^n = a+bx+cx^2+\ldots$, where a, b and c are real numbers and n is a positive integer.

(a) Find the value of a.

(b) If the coefficient of x^2 is -416, find the value(s) of n.

38. [Binomial Expansion, Exponential and Logarithmic functions]

(a) Let n be a positive integer. In the expansion of $(1+x)^{2n}+(1-x)^{2n}$, the coefficient of x^2 is 132. Find the value of n.

(b) Evaluate $1+C_2^{20}+C_4^{20}+\cdots+C_{20}^{20}$.

39. [Differentiation]

Figure B2-1 shows the graph of the first derivative of the function $y = f(x)$.

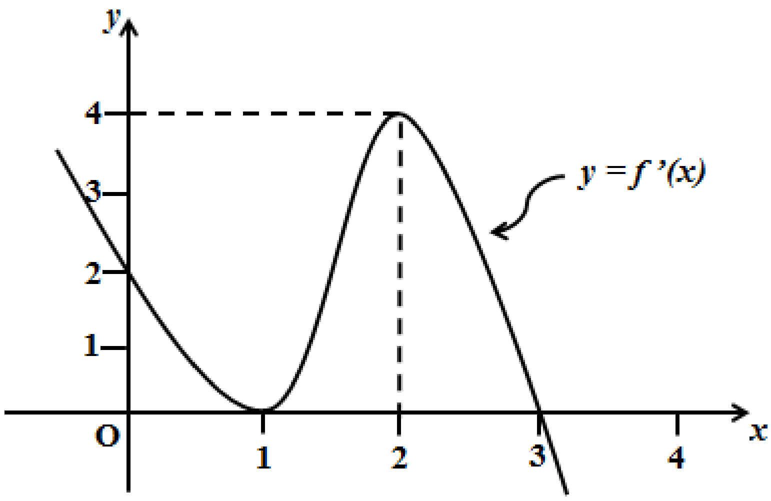


Figure B2-1

(a) If $x = x_0$ is a local maximum of $y = f(x)$, find the value of x_0.

(b) Find the range of value of x for which the graph of $y = f(x)$ is concave upwards.

(c) Suppose straight line L is tangent to the curve $y = f(x)$ at $x = 2$. If $f(2) = -1$, find the equation of L.

40. [Differentiation]

In Figure B2-2, a circle of radius 1 cm is shown, and O is its centre. P and Q are points on the circle. Let the height of $\triangle OPQ$ with PQ as its base be h cm.

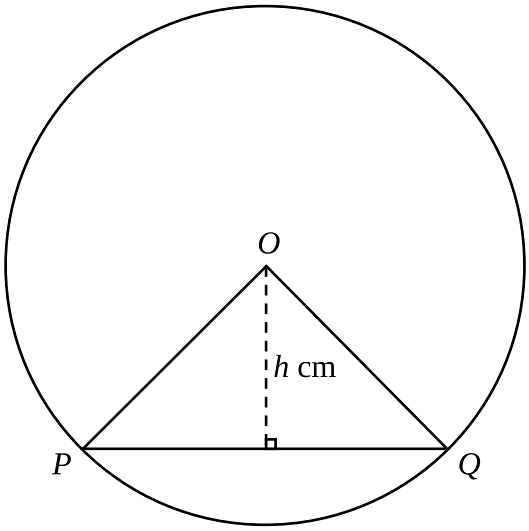


Figure B2-2

(a) (i) If the area of $\triangle OPQ$ is A cm^2, find A in terms of h.

(ii) Find $\angle POQ$ when the area of $\triangle OPQ$ attains its maximum.

(b) Vertices P and Q now move along the circle in such a way that the height of $\triangle OPQ$ (with PQ as base) increases at a rate of 3 cm per second.

(i) Find the rate of change of the length of PQ with respect to time when $h = 0.8$.

(ii) Find the rate of change of A with respect to time when $h = 0.8$.

41. [Integration]

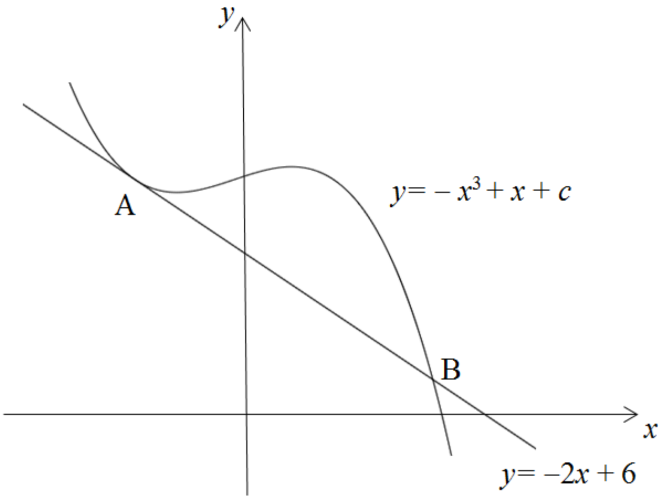


Figure B2-3

Figure B2-3 shows the graph of the curve $y=-x^3+x+c$. Straight line $y=-2x+6$ is tangent to the curve at *A* and cuts the curve at *B*. Find

(a) the value of *c*;

(b) the area of the region enclosed by $y=-x^3+x+c$ and $y=-2x+6$.

42. [Integration]

In Figure B2-4, *P*(*a*, *b*) is a point lying on the curve C : $y=8x^3+1$. From *P*, lines perpendicular to the coordinate axes are drawn to intersect the axes.

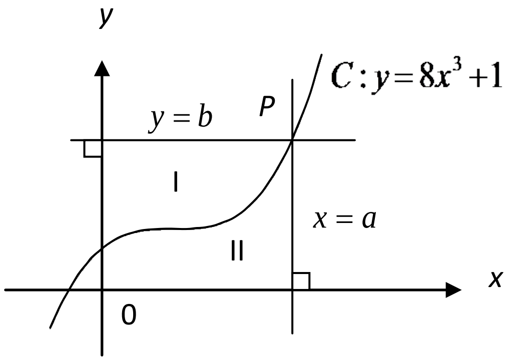


Figure B2-4

Region I is bounded by $y=b$, $y=8x^3+1$, and the *y*-axis. Region II is bounded by the *x*-axis, $y=8x^3+1$, the *y*-axis and straight line $x = a$.

(a) Find the area of Region I in terms of *a*.

(b) Hence, find the value of *a* if the areas of Region I and Region II are equal.

END OF SECTION B(2)

SECTION B(3)

Answer any FOUR questions in this section.

43. [Binomial expansion, Exponential and logarithmic functions]

(a) Let n be a positive integer. In the expansion of $(1+x)^{2n}+(1-x)^{2n}$, the coefficient of x^2 is 132. Find the value of n.

(b) Evaluate $1+C_2^{20}+C_4^{20}+\cdots+C_{20}^{20}$.

44. [Matrix]

(a) Among all 2 by 2 matrices whose entries are either 0s or 1s, how many of these matrices are invertible (non-singular)?

(b) It is given that $\det A=\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}=\alpha$. Find

$\det B=\begin{vmatrix} a+d & b+e & c+f \\ d+g & e+h & f+i \\ g+a & h+b & i+c \end{vmatrix}$ in terms of α.

45. [Differentiation]

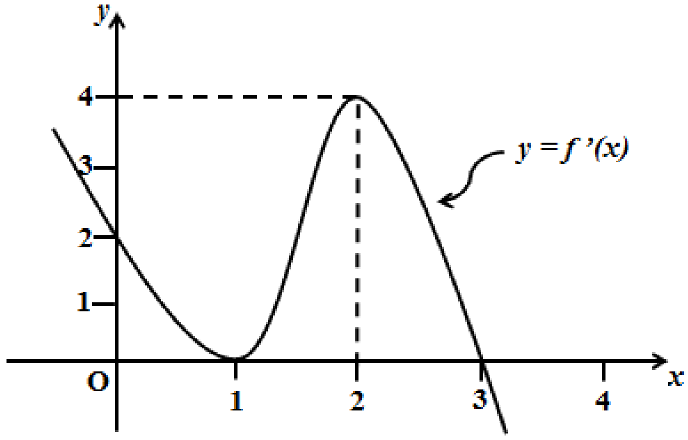


Figure B3-1

Figure B3-1 shows the graph of the first derivative of the function $y=f(x)$.

(a) If $x=x_0$ is a local maximum of $y=f(x)$, find the value of x_0.

(b) Find the range of value of x for which the graph of $y=f(x)$ is concave upwards.

(c) Suppose straight line L is the tangent to the curve $y=f(x)$ at $x=2$. If $f(2)=-1$, find the equation of L.

46. [Differentiation]

In Figure B3-2, a circle of radius 1 cm is shown, and O is its centre. P and Q are points on the circle. Let the height of $\triangle OPQ$ with PQ as its base be h cm.

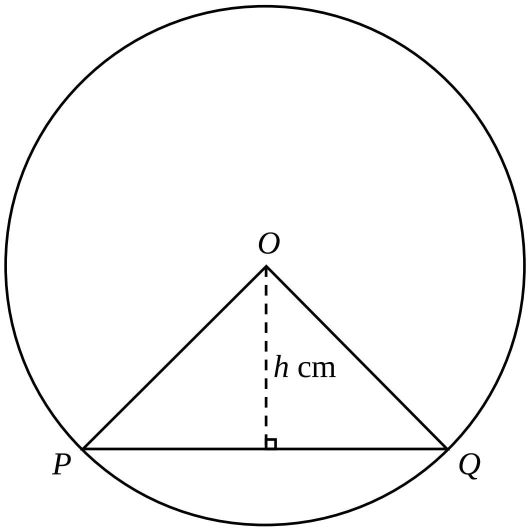


Figure B3-2

(a) (i) If the area of $\triangle OPQ$ is A cm^2, find A in terms of h.

(ii) Find $\angle POQ$ when the area of $\triangle OPQ$ attains its maximum.

(b) Vertices P and Q now move along the circle in such a way that the height of $\triangle OPQ$ (with PQ as base) increases at a rate of 3 cm per second.

(i) Find the rate of change of the length of PQ with respect to time when $h = 0.8$.

(ii) Find the rate of change of A with respect to time when $h = 0.8$.

47. [Integration]

A hole of radius 1 is drilled uniformly through the centre of a solid sphere of radius 2 as shown in Figure B3-3. Find the volume of material removed from the sphere. Express your answer in terms of π.

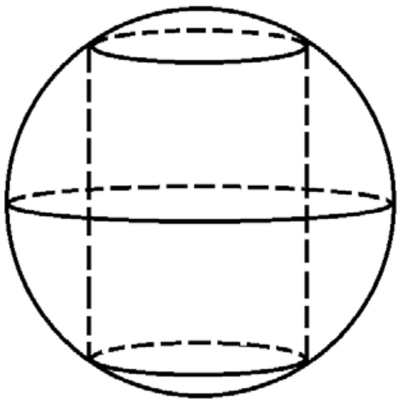


Figure B3-3

48. [Integration]

(a) It is given that $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{4}$. If $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,\mathrm{d}x + \pi = a$, where a is a rational number, find the value of a. Express your answer as a fraction.

(b) Using the inequality $\frac{x^4(1-x)^4}{2} < \frac{x^4(1-x)^4}{1+x^2} < \frac{x^4(1-x)^4}{1}$ for $0 < x < 1$ and the result of (a), find a lower limit and an upper limit of π. Correct your answers to 4 decimal places.

END OF SECTION B(3)

END OF PAPER